A MICROSCOPIC CONVEXITY PRINCIPLE FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

Caffarelli-Friedman [7] proved a *constant rank theorem* for convex solutions of semilinear elliptic equations in $\mathbb{R}^2$, a similar result was also discovered by Yau [28] at the same time. The result in [7] was generalized to $\mathbb{R}^n$ by Korevaar-Lewis [27] shortly after. This type of *constant rank theorem* is called microscopic convexity principle. It is a powerful tool in the study of geometric properties of solutions of nonlinear differential equations, it is particularly useful in producing convex solutions of differential equations via homotopic deformations. The great advantage of the microscopic convexity principle is that it can treat geometric nonlinear differential equations involving tensors on general manifolds. The proof of such microscopic convexity principle for σ_k-equation on the unit sphere $\mathbb{S}^n$ by Guan-Ma [17] is crucial in the study of the Christoffel-Minkowski problem. The microscopic convexity principle provides some interesting geometric properties of solutions to the equation. For symmetric Codazzi tensor, the microscopic convexity principle yields that the distribution of null space of the tensor is of constant dimension and it is parallel. The microscopic convexity principle has been validated for a varieties of fully nonlinear differential equations involving the second fundamental forms of hypersurfaces (e.g., [17, 16, 18, 8]).

Driven by the pertinent question that under what structural conditions for partial differential equations so that the microscopic convexity principle is held, Caffarelli-Guan-Ma [8] established such principle for the fully nonlinear equations of the form:

$$F(u_{ij}(x)) = \varphi(x, u(x), \nabla u(x)). \tag{1.1}$$

where $F(A)$ is a symmetric and $F(A^{-1})$ is locally convex in A. The similar results were also proved for symmetric tensors on manifolds in [8], along with several important geometric applications. It is important to consider equations where F involves other arguments in addition to the Hessian (u_{ij}). For example, it is desirable to include linear elliptic

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equations and quasilinear equations with variable coefficients. In many cases, a solution v to an equation itself may not be convex. Yet, some of its transformation may be convex (e.g., [6, 7]). If v is a solution of equation (1.1), $u = h(v)$ is a solution of equation

$$F(\nabla^2 u, \nabla u, u, x) = 0. \tag{1.2}$$

In general, $\nabla^2 u$ may not be separated from the rest of the arguments. The similar situation also arises in the case of geometric flow for hypersurfaces.

In this paper, we study the microscopic convexity property for equation in the form of (1.2) and related geometric nonlinear equations of elliptic and parabolic type. The core for the microscopic convexity principle is to establish a strong maximum principle for appropriate constructed functions. The key is to control ceratin gradient terms of the symmetric tensor to show that they are vanishing at the end. There have been significant development of analysis techniques in literature [7, 27, 17, 16, 18, 8] for this purpose, in particular the method introduced in [8]. They are very effective to control quadratic terms of the gradient of the symmetric tensor. For equation (1.2), linear terms of such gradient of symmetric tensor will emerge. All the previous methods break down for these terms. The main contribution of this paper is the introduction of new analytic techniques to handle these linear terms. This type new analysis involves quotients of elementary symmetric functions near the null set of $\det(u_{ij})$, even though equation (1.2) itself may not be symmetric with respect to the curvature tensor. The analysis is delicate and has to be balanced as both symmetric functions in the quotient will vanish at the null set. This is a novel feature of this paper, it is another indication that these quotient functions are naturally embedded with fully nonlinear equations. In a different context, the importance of quotient functions has been demonstrated in the beautiful work of Huisken-Sinestrari [22]. We believe the techniques in this paper will find way to solve other problems in geometric analysis.

To illustrate our main results, we first consider the equations in flat domain. Let Ω is a domain in $\mathbb{R}^n$, $\mathcal{S}^n$ denotes the space of real symmetric $n \times n$ matrices, and $F = F(r, p, u, x)$ is a given function in $\mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega$ and elliptic in the sense that

$$\left(\frac{\partial F}{\partial r_{\alpha\beta}}(\nabla^2 u, \nabla u, u, x)\right) > 0, \quad \forall x \in \Omega. \tag{1.3}$$

Theorem 1.1. *Suppose $F = F(r, p, u, x) \in C^{2,1}(\mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega)$ and F satisfies conditions (1.3) and*

$$F(A^{-1}, p, u, x) \quad \text{is locally convex in } (A, u, x) \text{ for each } p \text{ fixed.} \tag{1.4}$$

If $u \in C^{2,1}(\Omega)$ is a convex solution of (1.2), then the rank of Hessian $(\nabla^2 u(x))$ is constant l in Ω. For each $x_0 \in \Omega$, there exist a neighborhood $\mathcal{U}$ of x_0 and $(n-l)$ fixed directions $V_1, \cdots, V_{n-l}$ such that $\nabla^2 u(x) V_j = 0$ for all $1 \le j \le n-l$ and $x \in \mathcal{U}$.

There is also a parabolic version.

Theorem 1.2. *Suppose $F = F(r,p,u,x,t) \in C^{2,1}(\mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega \times [0,T))$ and F satisfies conditions (1.3) for each t and*

$$F(A^{-1}, p, u, x, t) \quad \text{is locally convex in } (A,u,x) \text{ for each } (p,t) \text{ fixed.} \tag{1.5}$$

Suppose $u \in C^{2,1}(\Omega \times [0,T))$ is a convex solution of the equation

$$\frac{\partial u}{\partial t} = F(\nabla^2 u, \nabla u, u, x, t). \tag{1.6}$$

For each $T > t > 0$, let $l(t)$ be the minimal rank of $(\nabla^2 u(x,t))$ in Ω. Then, the rank of $(\nabla^2 u(x,t))$ is constant for each $T > t > 0$ and $l(s) \le l(t)$ for all $s \le t < T$. For each $0 < t \le T$, $x_0 \in \Omega$, there exist a neighborhood $\mathcal{U}$ of x_0 and $(n - l(t))$ fixed directions $V_1, \cdots, V_{n-l(t)}$ such that $\nabla^2 u(x,t) V_j = 0$ for all $1 \le j \le n - l(t)$ and $x \in \mathcal{U}$. Furthermore, for any $t_0 \in [0,T)$, there is $\delta > 0$, such that the null space of $(\nabla^2 u(x,t))$ is parallel in (x,t) for all $x \in \Omega, t \in (t_0, t_0 + \delta)$.

An immediate consequence of Theorem 1.1 is the validation of a conjecture raised by Korevaar-Lewis in [27] for convex solutions of mean curvature type elliptic equation

$$\sum_{i,j} a^{ij}(\nabla u(x)) u_{ij}(x) = f(x, u(x), \nabla u(x)) > 0. \tag{1.7}$$

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Suppose u is a convex solution of elliptic equation (1.7). If*

$$\frac{1}{f(x,u,p)} \text{is locally convex in } (x,u) \text{ for each } p \text{ fixed,} \tag{1.8}$$

then the Hessian of u is of constant rank in Ω.

Korevaar-Lewis [27] proved that the Hessian of any convex solution u of elliptic equation (1.7) is of constant rank and u is constant in $n-l$ coordinate directions, provided that $\frac{1}{f(.,p)}$ is strictly convex for any p fixed. They conjectured that the constant rank result still holds if $\frac{1}{f(.,p)}$ is only assumed to be convex. They observed when $n=2$, this can be deduced from the proofs of Caffarelli-Friedman in [7]. Set

$$F(\nabla^2 u, \nabla u, u, x) = -\frac{1}{\sum_{i,j} a^{ij}(\nabla u(x)) u_{ij}(x)} + \frac{1}{f(x, u(x), \nabla u(x))},$$

Equation (1.7) is equivalent to $F(\nabla^2 u, \nabla u, u, x) = 0$. It is straightforward to check that F satisfies Conditions (1.3) and (1.4) under the assumptions in Corollary 1.3.

We now discuss some geometric equations on general manifolds. Preservation of convexity is an important issue for the geometric flows of hypersurfaces (e.g., [21, 5] and references therein). We have the following general result.

Theorem 1.4. *Suppose $F(A,X,\vec{n})$ is elliptic in A and $F(A^{-1},X,\vec{n})$ is locally convex in (A,X) for each fixed $\vec{n}\in\mathbb{S}^n$. Let $M(t)\subset\mathbb{R}^{n+1}$ be compact hypersurface and it is a solution of the geometric flow*

$$X_t = -F(g^{-1}h, X, \vec{n})\vec{n}, \quad t\in(0,T), \quad M(0)=M_0, \tag{1.9}$$

where $X,\vec{n},g,h$ are the position function, outer normal, induced metric and the second fundamental form of $M(t)$. If M_0 is convex, then $M(t)$ is strictly convex for all $t\in(0,T)$.

Alexandrov in [1, 3] studied existence and uniqueness of general nonlinear curvature equations,

$$F(g^{-1}h, X, \vec{n}(X)) = 0, \quad \forall X\in M, \tag{1.10}$$

where X is the position function of M and $\vec{n}(X)$ the unit normal of M at X. The following theorem addresses the convexity problems in [1, 3].

Theorem 1.5. *Suppose $F(A,X,\vec{n})$ is elliptic in A and $F(A^{-1},X,\vec{n})$ is locally convex in (A,X) for each fixed $\vec{n}\in\mathbb{S}^n$. Let M be an oriented immersed connect hypersurface in $\mathbb{R}^{n+1}$ with a nonnegative definite second fundamental form h satisfying equation (1.10), then h is of constant rank its null space is parallel. In particular, if M is complete, then there is $0\le l\le n$ such that $M=M^l\times\mathbb{R}^{n-l}$ for a strictly convex compact hypersurface M^l in $\mathbb{R}^{l+1}$. If in addition M is compact, then M is the boundary of a strongly convex bounded domain in $\mathbb{R}^{n+1}$.*

Theorem 1.5 shares some similarity with the classical result of Hartman-Nirenberg in [20].

The microscopic convexity principle can be used to prove some uniqueness theorems in differential geometry in large. An immersed surface in $\mathbb{R}^3$ is called Weingarten surface if its principle curvatures κ_1,κ_2 satisfy relationship $F(\kappa_1,\kappa_2)=0$ for some function F. Alexandrov [2] and Chern [12] proved that if M is a closed convex surface in $\mathbb{R}^3$ such that $F(\kappa_1,\kappa_2)=0$ for some elliptic F (i.e, F satisfies condition (1.3)), then M is a sphere. In higher dimensions, there is extensive literature devoted the sphere theorem of immersed hypersurfaces (e.g., [11, 13]). We prove the following sphere theorem, we refer to [17, 18, 8] for applications in classical and conformal geometry, and refer to [15] for applications in Kähler geometry.

Theorem 1.6. *Suppose (M, g) is a compact connected Riemannian manifold of dimension n with nonnegative sectional curvature, and positive at one point. Suppose $F(A)$ is elliptic, and W is a Codazzi tensor on M satisfying equation*

$$F(g^{-1}W) = 0 \quad \text{on } M. \tag{1.11}$$

If either

(1) *$n = 2$, or*
(2) *$n \geq 3$, W is semi-positive definite and $F(A^{-1})$ is locally convex for $A > 0$,*

then $W = cg$ for some constant $c \geq 0$.

Theorem 1.6 was proved by Ecker-Huisken in [13] under the assumption F is concave, we refer Remark 4.9 for relationship between concavity of $F(A)$ and condition on F in case (2) of Theorem 1.6. We note that when $n = 2$, only ellipticity assumption on F is needed in Theorem 1.6.

There is a vast literature devoted to the study of the convexity of solutions of partial differential equations. There is a theory of macroscopic nature, where problem is considered in a convex domain in $\mathbb{R}^n$ with proper boundary conditions. Korevaar made breakthroughs in [25, 26], he obtained concavity maximum principles for a class of quasi-linear elliptic equations defined convex domains in $\mathbb{R}^n$ in 1983. His results were improved by Kennington [24] and by Kawhol [23]. The theory further developed to its great generality by Alvarez-Lasry-Lions [4] in 1997, they established the existence of convex solution of equation (1.2) for state constraint boundary value under conditions (1.3)-(1.4) and that F satisfies comparison principle. Microscopic convexity implies macroscopic convexity if there is a deformation path (e.g., via the method of continuity or parabolic flow). Theorem 1.1 is the microscopic version of the macroscopic convexity principle in [4].

The rest of the paper is organized as follows. In section 2, we introduce a key auxiliary function $q(x)$ and derive certain negativity properties of this function (Proposition 2.1 and Corollary 2.2). In section 3, we establish a strong maximum principle for function $\phi(x) = \sigma_{l+1}(\nabla^2 u(x)) + q(x)$. In section 4, we discuss condition (1.4) and related results. The last section is devoted to geometric equations on manifolds.

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2. An Auxiliary function

To establish a microscopic convexity principle, one would like to prove the rank of $\nabla^2 u$ is of constant rank. It is natural to consider function $\sigma_{l+1}(\nabla^2 u)$ here l the minimal rank of $\nabla^2 u$. $\nabla^2 u$ is of constant rank is equivalent to $\sigma_{l+1}(\nabla^2 u) \equiv 0$. It was first shown by Caffarelli-Friedman in [7] that there is a strong maximum principle for $\sigma_{l+1}(\nabla^2 u)$ when $F = \Delta$ in $\mathbb{R}^2$. In the subsequential papers [27, 17, 16, 18], this type of maximum principle was establishes for differential functional F when it is either an elementary symmetric function of $\nabla^2 u$ or a quotient of them. In these papers, the analysis relies on the algebraic properties of the elementary symmetric functions. For general F in (1.1), the test function $\sigma_{l+1}(\nabla^2 u)$ was replaced by $\sigma_{l+1}(\nabla^2 u) + A\sigma_{l+2}(\nabla^2 u)$ (A large). All these are relied on one special fact: for symmetric function F in (1.1), all the third order derivatives (i.e., the gradient of the symmetric tensor $\nabla^2 u$) which appear in the process are always in quadratic order. This fact is important for above mentioned methods to work, we refer Remark 2.6 for a discussion of a unified argument.

When deal with general equation (1.2), linear terms of third order derivatives of u (i.e., the gradient of the symmetric tensor $\nabla^2 u$) will appear. How to control them is the major challenge. All the test functions considered before would yield certain "*good*" quadratic terms of third order derivatives which are **not strong** enough for this case, as linear terms can not be controlled by quadratic terms when they are assumed to be approaching 0 (we want prove all of them are vanishing at the end). We introduce a new auxiliary function which is composed as a quotient of elementary symmetric functions $\frac{\sigma_{l+2}(\nabla^2 u)}{\sigma_{l+1}(\nabla^2 u)}$ near points where $\nabla^2 u(x)$ is of minimal rank l. Though both $\sigma_{l+1}(\nabla^2 u)$ and $\sigma_{l+1}(\nabla^2 u)$ vanish at points where rank of $\nabla^2 u(x)$ is l, the Newton-MacLaurine inequality guarantee it is well defined. In fact, we will show $\frac{\sigma_{l+2}(\nabla^2 u)}{\sigma_{l+1}(\nabla^2 u)}$ has optimal $C^{1,1}$ regularity in Corollary 2.2. Furthermore, we will signal out some key concavity terms of this function in Proposition 2.1 to dominate the aforementioned linear terms of corresponding third order derivatives. The quotient function of elementary symmetric function plays a crucial role in this paper. We also call attention to the work of [22] for some other important roles of this type of functions in geometric analysis.

With the assumptions of F and u in Theorem 1.1 and Theorem 1.2, u is automatically in $C^{3,1}$. We will assume $u \in C^{3,1}(\Omega)$ in the rest of this paper. Let $W(x) = \nabla^2 u(x)$ and $l = \min_{x\in\Omega} \operatorname{rank}(\nabla^2 u(x))$. We may assume $l \le n-1$. Suppose $z_0 \in \Omega$ is a point where W is of minimal rank l.

Throughout this paper we assume that $\sigma_j(W) = 0$ if $j < 0$ or $j > n$. We define for $W = (u_{ij}) \in \mathcal{S}^n$

$$q(W) = \begin{cases} \frac{\sigma_{l+2}(W)}{\sigma_{l+1}(W)}, & \text{if } \sigma_{l+1}(W) > 0 \\ 0, & \text{if } \sigma_{l+1}(W) = 0 \end{cases} \tag{2.1}$$

For any symmetric function $f(W)$, we denote

$$f^{ij} = \frac{\partial f(W)}{\partial u_{ij}}, \quad f^{ij,km} = \frac{\partial^2 f(W)}{\partial u_{ij}\partial u_{km}}$$

For each $z_0 \in \Omega$ where W is of minimal rank l. We pick an open neighborhood $\mathcal{O}$ of z_0, for any $x \in \mathcal{O}$, let $\lambda_1(x) \le \lambda_2(x)... \le \lambda_n(x)$ be the eigenvalues of W at x. There is a positive constant $C > 0$ depending only on $\|u\|_{C^{3,1}}$, $W(z_0)$ and $\mathcal{O}$, such that $\lambda_n(x) \ge \lambda_{n-1}(x)... \ge \lambda_{n-l+1}(x) \ge C$ for all $x \in \mathcal{O}$. Let $G = \{n-l+1, n-l+2, ..., n\}$ and $B = \{1, ..., n-l\}$ be the "good" and "bad" sets of indices respectively. Let $\Lambda_G = (\lambda_{n-l+1}, ..., \lambda_n)$ be the "good" eigenvalues of W at x and $\Lambda_B = (\lambda_1, ..., \lambda_{n-l})$ be the "bad" eigenvalues of W at x. For the simplicity, we will also write $G = \Lambda_G$, $B = \Lambda_B$ if there is no confusion. Note that for any $\delta > 0$, we may choose $\mathcal{O}$ small enough such that $\lambda_i(x) < \delta$ for all $i \in B$ and $x \in \mathcal{O}$.

Set

$$\phi = \sigma_{l+1}(W) + q(W) \tag{2.2}$$

where q as in (2.1). We will use notation $h = O(f)$ if $|h(x)| \le Cf(x)$ for $x \in \mathcal{O}$ with positive constant C under control. It is clear that $\lambda_i = O(\phi)$ for all $i \in B$.

To get around $\sigma_{l+1}(W) = 0$, for $\epsilon > 0$ sufficient small, we consider

$$q_\epsilon(W) = \frac{\sigma_{l+2}(W_\epsilon)}{\sigma_{l+1}(W_\epsilon)}, \quad \phi_\epsilon(W) = \sigma_{l+1}(W_\epsilon) + q_\epsilon(W), \tag{2.3}$$

where $W_\epsilon = W + \epsilon I$. We will also denote $G_\epsilon = (\lambda_{n-l+1} + \epsilon, ..., \lambda_n + \epsilon)$, $B_\epsilon = (\lambda_1 + \epsilon, ..., \lambda_{n-1} + \epsilon)$

We will work on q_ϵ to obtain a uniform C^2 estimate independent of ϵ. One may also work directly on q at the points where $\sigma_{l+1}(\nabla^2 u) \ne 0$ to obtained the same results in the rest of this section (with all relative constants independent of chosen point). In any case, we prefer to work on q_ϵ.

Set

$$v(x) = u(x) + \frac{\epsilon}{2}|x|^2. \tag{2.4}$$

We have $W_\epsilon = (\nabla^2 v)$. To simplify the nations, we will write q for q_ϵ, W for W_ϵ, G for G_ϵ and B for B_ϵ with the understanding that all the estimates will be independent of ϵ. In

this setting, if we pick $\mathcal{O}$ small enough, there is $C > 0$ independent of ϵ such that

$$\sigma_{l+1}(W(x)) \geq C\epsilon, \quad \text{and} \quad \sigma_1(B(x)) \geq C\epsilon, \quad \text{for all } x \in \mathcal{O}. \tag{2.5}$$

The importance of the function q is reflected in the following proposition.

Proposition 2.1. *There are constants C_1, C_2 independent of ϵ such that at any point $z \in \mathcal{O}$ with W is diagonal, for any $\alpha, \beta \in \{1, \cdots, n\}$,*

$$\begin{aligned}\sum_{i,j,k,m} q^{ij,km} v_{ij\alpha} v_{km\beta} \leq C_1\phi + C_2 \sum_{i,j\in B} |\nabla v_{ij}| - 2 \sum_{i\in B, j\in G} \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)\lambda_j} v_{ij\alpha} v_{ji\beta} \\ - \frac{1}{\sigma_1^3(B)} \sum_{i\in B} (\sigma_1(B) v_{ii\alpha} - v_{ii} \sum_{j\in B} v_{jj\alpha})(\sigma_1(B) v_{ii\beta} - v_{ii} \sum_{j\in B} v_{jj\beta}) \\ - \frac{1}{\sigma_1(B)} \sum_{i,j\in B, i\neq j} v_{ij\alpha} v_{ji\beta} - \frac{2}{\sigma_1^3(B)} \sum_{i\in B} v_{ii} \sigma_1(B|i) v_{ii\alpha} v_{ii\beta}.\end{aligned} \tag{2.6}$$

The last three terms in (2.6) will play key role to dominate linear terms of $v_{ij\alpha}$ $(i, j \in B)$ in our proof of Theorem 1.1 in the next section.

Corollary 2.2. *Let $u \in C^{3,1}(\Omega)$ be a convex function and $W(x) = (u_{ij}(x)), x \in \Omega$. Let $l = \min_{x\in\Omega} \operatorname{rank}(W(x))$, then the function $q(x) = q(W(x))$ defined in (2.1) is in $C^{1,1}(\Omega)$.*

The rest of this section will be devoted to the proof of Proposition2.1, which involves some subtle analysis of function q. The proof of Corollary 2.2 will be given at the end of this section. In preparation, we will list several lemmas which are well known. For the sack of completeness, we will provide the proofs. Suppose W is any $n \times n$ diagonal matrix, we denote $(W|i)$ to be the $(n-1) \times (n-1)$ matrix with ith row and ith column deleted, and denote $(W|ij)$ to be the $(n-2) \times (n-2)$ matrix with i, jth rows and i, jth columns deleted.

Lemma 2.3. *Suppose W is diagonal. Then we have*

$$q^{ij} = \begin{cases} \frac{\sigma_{l+1}(W)\sigma_{l+1}(W|i) - \sigma_{l+2}(W)\sigma_l(W|i)}{\sigma_{l+1}^2(W)}, & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \textit{and}$$

(a). if $i = m, j = k, i \neq j$, then

$$q^{ij,km} = \frac{\sigma_l(W|ij)}{\sigma_{l+1}(W)} + \frac{\sigma_{l+2}(W)\sigma_{l-1}(W|ij)}{\sigma_{l+1}^2(W)}$$

(b). if $i = j = k = m$, then

$$q^{ij,km} = -2\frac{\sigma_l(W|i)}{\sigma_{l+1}^3(W)}[\sigma_{l+1}(W)\sigma_{l+1}(W|i) - \sigma_l(W|i)\sigma_{l+2}(W|i)]$$

(c). if $i=j, k=m, i\neq k$, then

$$q^{ij,km}=\frac{\sigma_l(W|ik)}{\sigma_{l+1}(W)}-\frac{\sigma_{l+1}(W|i)\sigma_l(W|k)}{\sigma_{l+1}^2(W)}-\frac{\sigma_{l+1}(W|k)\sigma_l(W|i)}{\sigma_{l+1}^2(W)}$$

$$-\frac{\sigma_{l+2}(W)\sigma_{l-1}(W|ik)}{\sigma_{l+1}^2(W)}+2\frac{\sigma_{l+2}(W)\sigma_l(W|i)\sigma_l(W|k)}{\sigma_{l+1}^3(W)}$$

(d). otherwise

$$q^{ij,km}=0$$

Proof. Since W is diagonal, it follows from Proposition 2.2 in [17]

$$\frac{\partial\sigma_\gamma(W)}{\partial v_{ij}}=\begin{cases}\sigma_{\gamma-1}(W|i), & \text{if } i=j\\ 0, & \text{if } i\neq j\end{cases}$$

and

$$\frac{\partial^2\sigma_\gamma(W)}{\partial v_{ij}\partial v_{km}}=\begin{cases}\sigma_{\gamma-2}(W|ik), & \text{if } i=j,k=m,i\neq k\\ -\sigma_{\gamma-2}(W|ij), & \text{if } i=m,j=k,i\neq j\\ 0, & \text{otherwise}\end{cases}$$

for $1\leq\gamma\leq n$. We obtain thus

$$\sigma_{l+1}^{ij}=\frac{\partial\sigma_{l+1}}{\partial W_{ij}}=\begin{cases}\sigma_l(W|i), & \text{if } i=j\\ 0, & \text{if } i\neq j\end{cases}$$

and

$$\sigma_{l+1}^{ij,km}=\frac{\partial^2\sigma_{l+1}}{\partial W_{ij}\partial W_{km}}=\begin{cases}\sigma_{l-1}(W|ik), & \text{if } i=j,k=m,i\neq k\\ -\sigma_{l-1}(W|ij) & \text{if } i=m,j=k,i\neq j\\ 0 & \text{otherwise}\end{cases}\tag{2.7}$$

A direct computation yields

$$q^{ij}=\frac{1}{\sigma_{l+1}(W)}\frac{\partial\sigma_{l+2}(W)}{\partial v_{ij}}-\frac{\sigma_{l+2}(W)}{\sigma_{l+1}^2(W)}\frac{\partial\sigma_{l+1}(W)}{\partial v_{ij}}\tag{2.8}$$

and

$$q^{ij,km}=\frac{1}{\sigma_{l+1}(W)}\frac{\partial^2\sigma_{l+2}(W)}{\partial v_{ij}\partial v_{km}}-\frac{1}{\sigma_{l+1}^2(W)}\frac{\partial\sigma_{l+2}(W)}{\partial v_{ij}}\frac{\partial\sigma_{l+1}(W)}{\partial v_{km}}$$

$$-\frac{1}{\sigma_{l+1}^2(W)}\frac{\partial\sigma_{l+2}(W)}{\partial v_{km}}\frac{\partial\sigma_{l+1}(W)}{\partial v_{ij}}-\frac{\sigma_{l+2}(W)}{\sigma_{l+1}^2(W)}\frac{\partial^2\sigma_{l+1}(W)}{\partial v_{ij}\partial v_{km}}$$

$$+2\frac{\sigma_{l+2}(W)}{\sigma_{l+1}^3(W)}\frac{\partial\sigma_{l+1}(W)}{\partial v_{ij}}\frac{\partial\sigma_{l+1}(W)}{\partial v_{km}}\tag{2.9}$$

The lemma follows from (2.8) and (2.9). □

Lemma 2.4. *Suppose W is diagonal, then*

$$q^{ij} = \begin{cases} \frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)} + O(\phi), & \text{if } i=j\in B \\ O(\phi), & \text{if } i=j\in G \\ 0, & \text{if } i\neq j. \end{cases}$$

Furthermore $q^{ij,km}$ can be computed as follows:

(1) *If $i,j,k,m\in G$,*

$$q^{ij,km}=O(\phi)$$

(2) *If $j\in G, i\in B$,*

$$q^{ji,ij}=q^{ij,ji}=-\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)v_{jj}}+O(\phi)$$

(3) *If $i,j\in B, i\neq j$,*

$$q^{ij,ji}=-\frac{1}{\sigma_1(B)}+O(1)$$

(4) *If $i\in B$,*

$$q^{ii,ii}=-\frac{2}{\sigma_1^3(B)}(\sigma_1(B)\sigma_1(B|i)-\sigma_2(B|i))+O(1)$$

(5) *If $i\in B, k\in G$,*

$$q^{kk,ii}=q^{ii,kk}=O(1)$$

(6) *If $i,k\in B, i\neq k$,*

$$q^{ii,kk}=\frac{2\sigma_2(B)-\sigma_1^2(B)+(v_{ii}+v_{kk})\sigma_1(B)}{\sigma_1^3(B)}+O(1)$$

(7) *otherwise*

$$q^{ij,km}=0.$$

Proof. From [17] we conclude that for $W=(G,B)$ and $\gamma\geq l$,

$$\sigma_\gamma(W)=\sum_{k=0}^{l}\sigma_k(G)\sigma_{\gamma-k}(B),$$

and

$$\sigma_\gamma(W|i)=\sum_{k=0}^{l}\sigma_k(G)\sigma_{\gamma-k}(B|i),\quad \text{for } i\in B;$$

$$\sigma_\gamma(W|i)=\sum_{k=0}^{l-1}\sigma_k(G|i)\sigma_{\gamma-k}(B),\quad \text{for } i\in G:$$

$$\sigma_\gamma(W|ij) = \sum_{k=0}^{l-2} \sigma_k(G|ij)\sigma_{\gamma-k}(B), \quad \text{ for } i,j \in G;$$

$$\sigma_\gamma(W|ij) = \sum_{k=0}^{l-1} \sigma_k(G|i)\sigma_{\gamma-k}(B|j), \quad \text{ for } i \in G, j \in B$$

$$\sigma_\gamma(W|ij) = \sum_{k=0}^{l} \sigma_k(G)\sigma_{\gamma-k}(B|ij), \quad \text{ for } i,j \in B,$$

where $\sigma_{\gamma-k}(B) = 0$ if $\gamma - k > n - l$. The lemma follows directly from lemma 2.3 and above formulae. $\square$

Next we establish an estimate for third order derivatives of convex functions.

Lemma 2.5. *Assume $u \in C^{3,1}(\Omega)$ is a convex function. Then there exists a positive constant C depending only on $dist\{\mathcal{O}, \partial\Omega\}$ and $\|v\|_{C^{3,1}(\Omega)}$ such that*

$$|v_{ij\alpha}(x)| \le C\Big(\sqrt{v_{ii}(x)} + \sqrt{v_{jj}(x)}\Big) \tag{2.10}$$

for all $x \in \mathcal{O}$ and $1 \le i, j, \alpha \le n$.

Proof. It follows from convexity of v that for any direction $\eta \in R^n$ with $|\eta| = 1$

$$v_{\eta\eta}(x) \ge 0$$

for all $x \in \Omega$. It's well known that for any nonnegative $C^{1,1}$ function h, $|\nabla h(x)| \le C h^{\frac{1}{2}}(x)$ for all $x \in \mathcal{O}$, where C depending only on $\|h\|_{C^{1,1}(\Omega)}$ and $dist\{\mathcal{O}, \partial\Omega\}$ (e.g., see [29]). We now infer

$$|v_{\eta\eta\alpha}(x)| \le C\sqrt{v_{\eta\eta}(x)}.$$

where C is a positive constant depending only on $dist\{\mathcal{O}, \partial\Omega\}$ and $\|v_{\eta\eta}\|_{C^{1,1}(\Omega)}$ (which can be controlled by $\|u\|_{C^{3,1}(\Omega)}$). Now set $\eta = i$ if $i = j$ and

$$\eta = \frac{1}{\sqrt{2}}(e_i + e_j) \quad \text{if} \quad i \ne j.$$

Proof of Lemma 2.5 is complete. $\square$

Remark 2.6. In [8], test function $\phi(x) = \sigma_{l+1}(\nabla^2 u(x)) + A\sigma_{l+2}(\nabla^2 u(x))$ was introduced. The term $A\sigma_{l+2}(\nabla^2 u(x))$ was used there to overcome quadratic terms of the third order derivatives. With Lemma 2.5, these quadratic terms of the third order derivatives in fact can be controlled by $\sigma_{l+1}(\nabla^2 u(x))$. Therefore, all the arguments in [8] can carry through for simpler test function $\phi(x) = \sigma_{l+1}(\nabla^2 u(x))$. Nevertheless, for general equation (1.2), we will see in the next section that linear terms of the third order derivatives will appear, the auxiliary function $q(x)$ will play crucial role to control these terms.

Proof of Proposition 2.1. Let us divide $\sum_{i,j,k,m} q^{ij,km} v_{ij\alpha} v_{km\beta}$ into three parts according to Lemma 2.3:

$$\sum_{i,j,k,m} q^{ij,km}(W(z)) v_{ij\alpha} v_{km\beta} = I_{\alpha\beta} + II_{\alpha\beta} + III_{\alpha\beta}, \tag{2.11}$$

where

$$I_{\alpha\beta} = \sum_{i \neq j} q^{ij,ji} v_{ij\alpha} v_{ji\beta},$$

$$II_{\alpha\beta} = \sum_{i=1}^{n} q^{ii,ii} v_{ii\alpha} v_{ii\beta}$$

and

$$III_{\alpha\beta} = \sum_{i \neq k} q^{ii,kk} v_{ii\alpha} v_{kk\beta}.$$

Lemma 2.4 yields

$$\begin{aligned} I_{\alpha\beta} &= \Big(\sum_{i,j \in G, i \neq j} + \sum_{i \in B, j \in G} + \sum_{j \in B, i \in G} + \sum_{i,j \in B, i \neq j}\Big) q^{ij,ji} v_{ij\alpha} v_{ji\beta} \\ &= O(\phi) + O\Big(\sum_{i,j \in B} |\nabla v_{ij}|\Big) - \frac{1}{\sigma_1(B)} \sum_{i,j \in B, i \neq j} v_{ij\alpha} v_{ji\beta} \\ &\quad -2 \sum_{i \in B, j \in G} \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B) v_{jj}} v_{ij\alpha} v_{ji\beta}. \end{aligned} \tag{2.12}$$

It follows that from Lemma 2.4

$$\begin{aligned} II_{\alpha\beta} &= \Big(\sum_{i \in G} + \sum_{i \in B}\Big) q^{ii,ii} v_{ii\alpha} v_{ii\beta} \\ &= O(\phi) + O\Big(\sum_{i,j \in B} |\nabla v_{ij}|\Big) - 2 \sum_{i \in B} \frac{\sigma_1(B)\sigma_1(B|i) - \sigma_2(B|i)}{\sigma_1^3(B)} v_{ii\alpha} v_{ii\beta} \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} III_{\alpha\beta} &= \Big(\sum_{i,j \in G, i \neq j} + \sum_{i \in B, j \in G} + \sum_{j \in B, i \in G} + \sum_{i,j \in B, i \neq j}\Big) q^{ii,jj} v_{ii\alpha} v_{jj\beta} \\ &= O(\phi) + O\Big(\sum_{i,j \in B} |\nabla v_{ij}|\Big) + \sum_{i \neq j, i,j \in B} \frac{2\sigma_2(B) - \sigma_1^2(B) + (v_{ii} + v_{jj})\sigma_1(B)}{\sigma_1^3(B)} v_{ii\alpha} v_{jj\beta}. \end{aligned} \tag{2.14}$$

By the identity, for any indices set A,

$$\begin{aligned}
&\sum_{i,j\in A, i\neq j}[2\sigma_2(A)-\sigma_1^2(A)+(v_{ii}+v_{jj})\sigma_1(A)]v_{ii\alpha}v_{jj\beta}\\
&\quad -2\sum_{i\in A}[\sigma_1(A)\sigma_1(A|i)-\sigma_2(A|i)]v_{ii\alpha}v_{ii\beta}\\
&= -\sum_{i\in A}(\sigma_1(A)v_{ii\alpha}-v_{ii}\sum_{j\in A}v_{jj\alpha})(\sigma_1(A)v_{ii\beta}-v_{ii}\sum_{j\in A}v_{jj\beta})\\
&\quad -2\sum_{i\in A}v_{ii}\sigma_1(A|i)v_{ii\alpha}v_{ii\beta}. \qquad (2.15)
\end{aligned}$$

In particular, setting $A=B$ in (2.15), we deduce

$$\begin{aligned}
II_{\alpha\beta}+III_{\alpha\beta} &= O(\phi)+O(\sum_{i,j\in B}|\nabla v_{ij}|)-\frac{2}{\sigma_1^3(B)}\sum_{i\in B}v_{ii}\sigma_1(B|i)v_{ii\alpha}v_{ii\beta}\\
&- \frac{1}{\sigma_1^3(B)}\sum_{i\in B}(\sigma_1(B)v_{ii\alpha}-v_{ii}\sum_{j\in B}v_{jj\alpha})(\sigma_1(B)v_{ii\beta}-v_{ii}\sum_{j\in B}v_{jj\beta}). \qquad (2.16)
\end{aligned}$$

□

Finally, we prove Corollary 2.2.

Proof of Corollary 2.2. We only need to consider a small neighborhood $\mathcal{O}$ of these point $p\in\Omega$ such that the minimal rank is attained at p. For such fixed point $z\in\mathcal{O}$, we may assume $W(z)$ is diagonal by a rotation. We thus obtain for any fixed α and β

$$\frac{\partial^2 q(z)}{\partial x_\alpha\partial x_\beta}=\sum_{i,j}q^{ij}(W(z))u_{ij\alpha\beta}+\sum_{i,j,k,m}q^{ij,km}(W(z))u_{ij\alpha}u_{km\beta} \qquad (2.17)$$

Since $0\le\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)}\le 1$, by Lemma 2.4

$$|q^{ij}(W(z))|\le C$$

for some constant C under control. It yields the estimate for the first term in (2.17)

$$\|q^{ij}(W(z))u_{ij\alpha\beta}\|\le C\|u\|_{C^{3,1}(\Omega)}\le C$$

We treat the second term in (2.17). By Lemma 2.5, for $i,j\in B$

$$|u_{ij\alpha}|\le C(\sqrt{u_{ii}(x)}+\sqrt{u_{jj}(x)})\le C\sqrt{\sigma_1(B)}. \qquad (2.18)$$

Noting that $u_{jj}\ge C>0, j\in G$ and $0\le\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)}\le 1$. It now follows from Proposition 2.1,

$$|\frac{\partial^2 q(W(z))}{\partial x_\alpha\partial x_\beta}|\le C$$

for all $z\in\mathcal{O}$. □

3. A strong maximum principle

In this section, we prove a strong maximum principle for ϕ defined in (2.2) for equation (1.2). We may prove the same result for equation (1.6) and make Theorem 1.1 as a corollary of Theorem 1.2. But we prefer to work on elliptic case first. The parabolic version will be proved at the end of next section with some minor modification.

We denote $\mathcal{S}^n$ to be the set of all real symmetric $n\times n$ matrices, and denote $\mathcal{S}^n_+ \subset \mathcal{S}^n$ to be the set of all positive definite symmetric $n\times n$ matrices. Let $\mathbb{O}_n$ be the space consisting all $n\times n$ orthogonal matrices. We define

$$\mathcal{S}_{n-1} = \{Q\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}Q^T \quad | \quad \forall Q\in\mathbb{O}_n, \forall B\in\mathcal{S}^{n-1}\ \},$$

and for given $Q\in\mathbb{O}_n$,

$$\mathcal{S}_{n-1}(Q) = \{Q\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}Q^T \quad | \quad \forall B\in\mathcal{S}^{n-1}\ \}.$$

Therefore $\mathcal{S}_{n-1}, \mathcal{S}_{n-1}(Q)\subset\mathcal{S}^n$. For any function $F(r,p,u,x)$, we denote

$$F^{\alpha\beta} = \frac{\partial F}{\partial r_{\alpha\beta}},\quad F^u = \frac{\partial F}{\partial u},\quad F^{x_i} = \frac{\partial F}{\partial x_i},\quad F^{\alpha\beta,\gamma\eta} = \frac{\partial^2 F}{\partial r_{\alpha\beta}\partial r_{\gamma\eta}},\quad F^{\alpha\beta,u} = \frac{\partial^2 F}{\partial r_{\alpha\beta}\partial u},$$

$$F^{\alpha\beta,x_k} = \frac{\partial^2 F}{\partial r_{\alpha\beta}\partial x_k},\quad F^{u,u} = \frac{\partial^2 F}{\partial^2 u},\quad F^{u,x_i} = \frac{\partial^2 F}{\partial u\partial x_i},\quad F^{x_i,x_j} = \frac{\partial^2 F}{\partial x_i\partial x_j}. \tag{3.1}$$

For any p fixed and $Q\in\mathbb{O}_n$, $(A,u,x)\in\mathcal{S}_{n-1}(Q)\times\mathbb{R}\times\mathbb{R}^n$, we set

$$X^*_F = ((F^{\alpha\beta}(A,p,u,x)), -F^u(A,p,u,x), -F^{x_1}(A,p,u,x),\cdots,-F^{x_1}(A,p,u,x))$$

as a vector in $\mathcal{S}^n\times\mathbb{R}\times\mathbb{R}^n$. Set

$$\Gamma^{\perp}_{X^*_F} = \{\tilde X\in\mathcal{S}_{n-1}(Q)\times\mathbb{R}\times\mathbb{R}^n \quad | \quad <\tilde X, X^*_F>=0\}, \tag{3.2}$$

Let $B\in\mathcal{S}^{n-1}_+, A=B^{-1}$ and

$$\tilde B = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix},\quad \tilde A = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

For any given $Q\in\mathbb{O}_n$ and $\tilde X = ((X_{ij}), Y, Z_1,\cdots,Z_n)\in\mathcal{S}_{n-1}(Q)\times\mathbb{R}\times\mathbb{R}^n$, we define a quadratic form

$$\begin{aligned} Q^*(\tilde X,\tilde X) \quad = \quad & \sum_{i,j,k,l=1}^n F^{ij,kl}X_{ij}X_{kl} + 2\sum_{i,j,k,l=1}^n F^{ij}(Q\tilde A Q^T)_{kl}X_{ik}X_{jl} + \sum_{i,j=1}^n F^{x_i,x_j}Z_iZ_j \\ & -2\sum_{i,j=1}^n F^{ij,u}X_{ij}Y - 2\sum_{i,j,k=1}^n F^{ij,x_k}X_{ij}Z_k + 2\sum_{i=1}^n F^{u,x_i}YZ_i + F^{u,u}Y^2, \end{aligned} \tag{3.3}$$

where functions $F^{ij,kl}, F^{ij}, F^{u,u}, F^{ij,u}, F^{ij,x_k}, F^{u,x_i}, F^{x_i,x_j}$ are evaluated at $(Q\tilde BQ^T, p, u, x)$.

We first state a lemma, it's proof will be given in next section (after Corollary 4.2).

Lemma 3.1. *If F satisfies condition (1.4), then for each $p \in \mathbb{R}^n$,*

$$F(0,p,u,x) \quad \text{is locally convex in } (u,x), \text{ and } Q^*(\tilde{X},\tilde{X}) \geq 0, \forall \tilde{X} \in \Gamma^{\perp}_{X_F^*}. \tag{3.4}$$

The following theorem is the core of this paper. Theorem 1.1 is a direct consequence of Theorem 3.2 and Lemma 3.1.

Theorem 3.2. *Suppose that the function F satisfies conditions (1.3) and (3.4), let $u \in C^{3,1}(\Omega)$ is a convex solution of (1.2). If $\nabla^2 u$ attains minimum rank l at certain point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O}$ of x_0 and a positive constant C independent of ϕ (defined in (2.2)), such that*

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta}(x) \leq C(\phi(x) + |\nabla\phi(x)|), \quad \forall x \in \mathcal{O}. \tag{3.5}$$

In turn, $\nabla^2 u$ is of constant rank in $\mathcal{O}$. Moreover, for each $x_0 \in \Omega$, there exist a neighborhood $\mathcal{U}$ of x_0 and $(n-l)$ fixed directions $V_1, \cdots, V_{n-l}$ such that $\nabla^2 u(x)V_j = 0$ for all $1 \leq j \leq n-l$ and $x \in \mathcal{U}$.

Proof of Theorem 3.2. Let $u \in C^{3,1}(\Omega)$ be a convex solution of equation (1.2) and $W(x) = (u_{ij}(x))$. For each $z_0 \in \Omega$ where $W = (\nabla^2 u)$ attains minimal rank l. We may assume $l \leq n-1$, otherwise there is nothing to prove. As in the previous section, we pick an open neighborhood $\mathcal{O}$ of z_0, for any $x \in \mathcal{O}$, let $G = \{n-l+1, n-l+2, ..., n\}$ and $B = \{1, ..., n-l\}$ be the "good" and "bad" sets of indices for eigenvalues of $\nabla^2 u(x)$ respectively.

Setting ϕ as (2.2), then we see from Corollary 2.2 that $\phi \in C^{1,1}(\mathcal{O})$,

$$\phi(x) \geq 0, \ \phi(z_0) = 0$$

and there is a constant $C > 0$ such that for all $x \in \mathcal{O}$,

$$\frac{1}{C}\sigma_1(B)(x) \leq \phi(x) \leq C\sigma_1(B)(x), \ \frac{1}{C}\sigma_1(B)(x) \leq \sigma_{l+1}(x) \leq C\sigma_1(B)(x).$$

We shall fix a point $z \in \mathcal{O}$ and prove (3.5) at z. For each $z \in \mathcal{O}$ fixed, letting $\lambda_1 \leq \lambda_2 ... \leq \lambda_n$ be the eigenvalues of $W(z) = (u_{ij}(z))$ at z, we can rotate coordinate so that $W(z) = (u_{ij}(z))$ is diagonal, and $u_{ii}(z) = \lambda_i, i = 1, \cdots, n$. We note that all quantities involving g, q and ϕ are invariant under rotation.

Again, as in the previous section, we will avoid to deal with $\sigma_{l+1}(W) = 0$ by considering for W_ϵ (defined in (2.3)) for $\epsilon > 0$ sufficient small, with $W_\epsilon = W + \epsilon I$, $G_\epsilon = (\lambda_{n-l+1} + \epsilon, ..., \lambda_n + \epsilon)$, $B_\epsilon = (\lambda_1 + \epsilon, ..., \lambda_{n-1} + \epsilon)$. We note that W_ϵ is the Hessian of function $u_\epsilon(x) = u(x) + \frac{\epsilon}{2}|x|^2$. This function $u_\epsilon(x)$ satisfies equation

$$F(\nabla^2 u_\epsilon, \nabla u_\epsilon, u_\epsilon, x) = R_\epsilon, \tag{3.6}$$

where $R_\epsilon(x) = F(\nabla^2 u_\epsilon, \nabla u_\epsilon, u_\epsilon, x) - F(\nabla^2 u, \nabla u, u, x)$. Since $u \in C^{3,1}$, we have

$$|R_\epsilon(x)| \le C\epsilon, \quad |\nabla R_\epsilon(x)| \le C\epsilon, \quad |\nabla^2 R_\epsilon(x)| \le C\epsilon, \quad \forall x \in \mathcal{O}. \tag{3.7}$$

We will work on equation (3.6) to obtain differential inequality (3.5) for ϕ_ϵ defined in (2.3) with constant C_1, C_2 independent of ϵ. Theorem 3.2 would follow by letting $\epsilon \to 0$.

Set $v = u_\epsilon$, in the rest of this section, we will write W for W_ϵ, G for G_ϵ, B for B_ϵ, q for q_ϵ and ϕ for ϕ_ϵ, with the understanding that all the estimates will be independent of ϵ. We note that by (2.5), we have

$$\epsilon \le C\phi(x), \quad \text{for all } x \in \mathcal{O}, \tag{3.8}$$

and v satisfies equation

$$F(\nabla^2 v, \nabla v, v, x) = R(x), \tag{3.9}$$

with $R(x)$ under control as follows,

$$|\nabla^j R(x)| \le C\phi(x), \quad \text{for all } j = 0, 1, 2, \quad \text{and for all } x \in \mathcal{O}. \tag{3.10}$$

Simple computation yields

$$\phi_\alpha = \frac{\partial \phi}{\partial x_\alpha} = \phi^{ij} v_{ij\alpha}, \quad \phi_{\alpha\beta} = \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta} = \phi^{ij} v_{ij\alpha\beta} + \phi^{ij,km} v_{ij\alpha} v_{km\beta}.$$

We differentiate equation (3.9) in x_i, by (3.10),

$$\sum_{\alpha\beta} F^{\alpha\beta} v_{\alpha\beta i} + \sum_k F^{q_k} v_{ki} + F^v v_i + F^{x_i} = O(\phi), \tag{3.11}$$

and differentiate equation (3.9) twice with respect to the variables x_i and x_j, again by (3.10),

$$\begin{aligned}
&\sum_{\alpha\beta} F^{\alpha\beta} v_{\alpha\beta ij} + \sum_{\alpha\beta} v_{\alpha\beta i}\Big(\sum_{\gamma\eta} F^{\alpha\beta,\gamma\eta} v_{\gamma\eta j} + \sum_k F^{\alpha\beta,q_k} v_{kj} + F^{\alpha\beta,v} v_j + F^{\alpha\beta,x_j}\Big) \\
&+ \sum_k F^{q_k} v_{kij} + \sum_{k\alpha\beta} v_{ki}\Big(\sum_{\alpha\beta} F^{q_k,\alpha\beta} v_{\alpha\beta j} + \sum_l F^{q_k,q_l} v_{lj} + F^{q_k,v} v_j + F^{q_k,x_j}\Big) \\
&+ F^v v_{ij} + v_i\Big(\sum_{\alpha\beta} F^{v,\alpha\beta} v_{\alpha\beta j} + \sum_l F^{v,q_l} u_{lj} + F^{v,v} v_j + F^{v,x_j}\Big) \\
&+ \sum_{\alpha\beta} F^{x_i,\alpha\beta} v_{\alpha\beta j} + \sum_k F^{x_i,q_k} v_{kj} + F^{x_i,v} v_j + F^{x_i,x_j} = O(\phi).
\end{aligned} \tag{3.12}$$

As $v_{\alpha\beta ij} = v_{ij\alpha\beta}$ (this fact will have to be modified later by a commutator formula when we deal with symmetric curvature tensors on general manifolds), we get

$$\begin{aligned}
\sum F^{\alpha\beta}\phi_{\alpha\beta} &= \sum F^{\alpha\beta}\phi^{ij}v_{ij\alpha\beta} + \sum F^{\alpha\beta}\phi^{ij,km}v_{ij\alpha}v_{km\beta} \\
&= \sum F^{\alpha\beta}\phi^{ij,km}v_{ij\alpha}v_{km\beta} - \sum \phi^{ij}F^{q_k}v_{kij} \\
&\quad - \sum \phi^{ij}[F^v v_{ij} + 2\sum F^{\alpha\beta,q_k}v_{\alpha\beta i}v_{kj} + \sum F^{q_k,q_l}v_{ki}v_{lj} \\
&\quad + 2\sum F^{q_k,v}v_{ki}v_j + 2\sum F^{q_k,x_j}v_{ki}] \\
&\quad - \sum \phi^{ij}[F^{\alpha\beta,\gamma\eta}v_{\alpha\beta i}v_{\gamma\eta j} + 2\sum F^{\alpha\beta,v}v_{\alpha\beta i}v_j + 2\sum F^{\alpha\beta,x_j}v_{\alpha\beta i} \\
&\quad + \sum F^{v,v}v_i v_j + 2\sum F^{v,x_j}v_j + \sum F^{x_i x_j}] + O(\phi) \qquad (3.13)
\end{aligned}$$

We will deal terms in the right hand side of (3.13). The basic idea is to regroup them according indices in G and B. The analysis will be devoted to those third order derivatives terms which have with at least two indices in B. Since it contains some linear terms of such third order derivatives, previous arguments in [8] are not suitable here. The introduction of function q in (2.1) is the key, the concavity results of q in last section will be used in crucial way. As for the rest terms left in (3.13), we will sort them out in a way such that condition (1.4) can be used to obtain appropriate control.

We note that since $W = (v_{ij})$ is diagonal at z, by Lemma 2.3 and Lemma 2.4,

$$\phi^{ij}(z) = \begin{cases} \sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)} + O(\phi), & \text{if } i = j \in B \\ O(\phi), & \text{if } i = j \in G \\ 0, & \text{if } i \neq j \end{cases} \qquad (3.14)$$

Hence at z

$$\begin{aligned}
&\sum_{i,j}\phi^{ij}[F^v v_{ij} + 2\sum F^{\alpha\beta,q_k}v_{\alpha\beta i}v_{kj} + \sum F^{q_k,q_l}v_{ki}v_{lj} + 2\sum(F^{q_k,v}v_{ki}v_j + F^{q_k,x_j}v_{ki})] \\
&= \sum_{i=1}^{n}\phi^{ii}[F^v v_{ii} + 2\sum F^{\alpha\beta,q_i}v_{\alpha\beta i}v_{ii} + F^{q_i,q_i}v_{ii}v_{ii} + 2F^{q_i,v}v_{ii}v_i + 2F^{q_i,x_i}v_{ii}] \\
&= O(\phi) + \sum_{i\in B}\phi^{ii}[F^v + 2\sum F^{\alpha\beta,q_i}v_{\alpha\beta i} + F^{q_i,q_i}v_{ii} + 2F^{q_i,v}v_i + 2F^{q_i,x_i}]v_{ii} \\
&\leq O(\phi) + C\sum_{i\in B}(\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)})v_{ii} = O(\phi). \qquad (3.15)
\end{aligned}$$

This takes care of the third term in the right hand side of (3.13). For the second term there, we have

$$\sum \phi^{ij}F^{q_k}v_{kij} = O(\phi) + \sum_{i\in B}\phi^{ii}F^{q_k}v_{kii} = O(\phi + \sum_{i,j\in B}|\nabla v_{ij}|) \qquad (3.16)$$

For the fourth term in (3.13), by (3.14) we have,

$$
\begin{aligned}
&\phi^{ij}[F^{\alpha\beta,\gamma\eta}v_{\alpha\beta i}v_{\gamma\eta j}+2F^{\alpha\beta,v}v_{\alpha\beta i}v_j+2F^{\alpha\beta,x_j}v_{\alpha\beta i}+F^{v,v}v_iv_j+2F^{v,x_j}v_j+F^{x_ix_j}]\\
&=O(\phi)+\sum_{i\in B}\phi^{ii}[\sum F^{\alpha\beta,\gamma\eta}v_{\alpha\beta i}v_{\gamma\eta i}+2\sum F^{\alpha\beta,v}v_{\alpha\beta i}v_i\\
&\quad+2\sum F^{\alpha\beta,x_i}v_{\alpha\beta i}+F^{v,v}v_i^2+2F^{v,x_i}v_i+F^{x_ix_i}]\\
&=O(\phi+\sum_{i,j\in B}|\nabla v_{ij}|)+\sum_{i\in B}(\sigma_l(G)+\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)})\\
&\quad[\sum_{\alpha,\beta,\gamma,\eta\in G}F^{\alpha\beta,\gamma\eta}v_{i\alpha\beta}v_{i\gamma\eta}+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,v}v_{i\alpha\beta}v_i+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,x_i}v_{i\alpha\beta}\\
&\quad+F^{v,v}v_i^2+2F^{v,x_i}v_i+F^{x_ix_i}].
\end{aligned}\tag{3.17}
$$

Now we deal with the term $\sum F^{\alpha\beta}\phi^{ij,km}v_{ij\alpha}v_{km\beta}$ in (3.13). We note that

$$\phi^{ij,km}=\sigma_{l+1}^{ij,km}+q^{ij,km}.$$

Since $\sigma_{l-1}(W|ij)=O(\phi)$ for $i,j\in G, i\neq j$, for α,β fixed, by (2.7),

$$
\begin{aligned}
\sum\sigma_{l+1}^{ij,km}v_{ij\alpha}v_{km\beta} &= \sum_{i\neq k}\sigma_{l+1}^{ii,kk}v_{ii\alpha}v_{kk\beta}+\sum_{i\neq j}\sigma_{l+1}^{ij,ji}v_{ij\alpha}v_{ji\beta}\\
&= \sum_{i\neq k}\sigma_{l-1}(W|ik)v_{ii\alpha}v_{kk\beta}-\sum_{i\neq j}\sigma_{l-1}(W|ij)v_{ij\alpha}v_{ji\beta}\\
&= O(\phi+\sum_{i,j\in B}|\nabla v_{ij}|)-2\sum_{i\in B,j\in G}\sigma_{l-1}(G|j)v_{ij\alpha}v_{ij\beta}.
\end{aligned}
$$

As $\sigma_{l-1}(G|j)=\frac{\sigma_l(G)}{\lambda_j}, j\in G$, we have

$$\sigma_{l+1}^{ij,km}v_{ij\alpha}v_{km\beta}=O(\phi+\sum_{i,j\in B}|\nabla v_{ij}|)-2\sigma_l(G)\sum_{i\in B,j\in G}\frac{1}{\lambda_j}v_{ij\alpha}v_{ij\beta}.$$

By Proposition 2.1,

$$
\begin{aligned}
&\sum_{i,j,k,m}q^{ij,km}v_{ij\alpha}v_{km\beta}=O(\phi+\sum_{i,j\in B}|\nabla v_{ij}|)-2\sum_{i\in B,j\in G}\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)\lambda_j}v_{ij\alpha}v_{ji\beta}\\
&\quad-\frac{1}{\sigma_1^3(B)}\sum_{i\in B}(\sigma_1(B)v_{ii\alpha}-v_{ii}\sum_{j\in B}v_{jj\alpha})(\sigma_1(B)v_{ii\beta}-v_{ii}\sum_{j\in B}v_{jj\beta})\\
&\quad-\frac{1}{\sigma_1(B)}\sum_{i,j\in B,i\neq j}v_{ij\alpha}v_{ji\beta}-\frac{2}{\sigma_1^3(B)}\sum_{i\in B}v_{ii}\sigma_1(B|i)v_{ii\alpha}v_{ii\beta}.
\end{aligned}
$$

We conclude that

$$\begin{aligned}\sum F^{\alpha\beta}\phi^{ij,km}v_{ij\alpha}v_{km\beta} &= O(\phi+\sum_{i,j\in B}|\nabla v_{ij}|)-\sum_{\alpha,\beta}F^{\alpha\beta}[\frac{2\sum_{i\in B}v_{ii}\sigma_1(B|i)v_{ii\alpha}v_{ii\beta}}{\sigma_1^3(B)}\\ &-\frac{1}{\sigma_1(B)}\sum_{i,j\in B,i\neq j}v_{ij\alpha}v_{ji\beta}-2\sum_{i\in B}(\sigma_l(G)+\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)})\frac{1}{\lambda_j}v_{ij\alpha}v_{ji\beta}\\ &-\frac{1}{\sigma_1^3(B)}\sum_{i\in B}(\sigma_1(B)v_{ii\alpha}-v_{ii}\sum_{j\in B}v_{jj\alpha})(\sigma_1(B)v_{ii\beta}-v_{ii}\sum_{j\in B}v_{jj\beta})].\end{aligned}\tag{3.18}$$

Combining (3.15)-(3.18), (3.13) is deduced to

$$\begin{aligned}\sum F^{\alpha\beta}\phi_{\alpha\beta} &= O(\phi+\sum_{i,j\in B}|\nabla v_{ij}|)-\frac{1}{\sigma_1(B)}\sum_{\alpha,\beta}\sum_{i,j\in B,i\neq j}F^{\alpha\beta}v_{ij\alpha}v_{ij\beta}\\ &-\frac{2}{\sigma_1^3(B)}\sum_{\alpha,\beta}\sum_{i\in B}F^{\alpha\beta}v_{ii}\sigma_1(B|i)v_{ii\alpha}v_{ii\beta}\\ &-\frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta}\sum_{i\in B}F^{\alpha\beta}(v_{ii\alpha}\sigma_1(B)-v_{ii}\sum_{j\in B}v_{jj\alpha})(v_{ii\beta}\sigma_1(B)-v_{ii}\sum_{j\in B}v_{jj\beta})\\ &-\sum_{i\in B}[\sigma_l(G)+\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)}][\sum_{\alpha,\beta,\gamma,\eta\in G}F^{\alpha\beta,\gamma\eta}(\Lambda)v_{i\alpha\beta}v_{i\gamma\eta}\\ &+2\sum_{\alpha\beta\in G}F^{\alpha\beta}\sum_{j\in G}\frac{1}{\lambda_j}v_{ij\alpha}v_{ij\beta}+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,v}v_{i\alpha\beta}v_i\\ &+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,x_i}v_{i\alpha\beta}+F^{v,v}v_i^2+2F^{v,x_i}v_i+F^{x_i,x_i}].\end{aligned}\tag{3.19}$$

At this point, we have succeeded in regrouping of terms involving third order derivatives. We first estimate the fifth term on the right hand side of (3.19). For each $i\in B$, let

$$\begin{aligned}J_i &= [\sum_{\alpha,\beta,\gamma,\eta\in G}F^{\alpha\beta,\gamma\eta}v_{i\alpha\beta}v_{i\gamma\eta}+2\sum_{\alpha,\beta\in G}F^{\alpha\beta}\sum_{j\in G}\frac{1}{\lambda_j}v_{ij\alpha}v_{ij\beta}\\ &\quad+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,v}v_{i\alpha\beta}v_i+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,x_i}v_{i\alpha\beta}+F^{v,v}v_i^2+2F^{v,x_i}v_i+F^{x_i,x_i}].\end{aligned}\tag{3.20}$$

If $l=0$, then $G=\emptyset$ and

$$J_i=F^{v,v}(\nabla^2v,\nabla v,v,z)v_i^2+2F^{v,x_i}(\nabla^2v,\nabla v,v,z)v_i+F^{x_i,x_i}(\nabla^2v,\nabla v,v,z).$$

Since $F\in C^{2,1}$ and $|\nabla^2v(z)|=O(\phi)$, by condition (3.4),

$$J_i=F^{v,v}(0,\nabla v,v,z)v_i^2+2F^{v,x_i}(0,\nabla v,v,z)v_i+F^{x_i,x_i}(0,\nabla v,v,z)+O(\phi)\geq -C\phi.$$

We may assume $1 \le l \le n-1$. By Condition (1.3), since $v \in C^{3,1}$ so $F^{\alpha\beta} \in C^{0,1}$, as $\bar{\mathcal{O}} \subset \Omega$, there exists a constant $\delta_0 > 0$, such that

$$(F^{\alpha\beta}) \geq \delta_0 I, \ \forall y \in \mathcal{O}. \tag{3.21}$$

As $l \geq 1$, so $n \in G$ and $F^{nn} \geq \delta_0$. From (3.11), since $v_{ik} = \delta_{ik}\lambda_i$ at z, we have for $i \in B$

$$\sum_{\alpha,\beta\in G} F^{\alpha\beta} v_{\alpha\beta i} + F^v v_i + F^{x_i} = O(\phi + \sum_{i,j\in B} |\nabla v_{ij}|),$$

Now let's set $X_{\alpha\beta} = 0$, $\alpha \in B$ or $\beta \in B$,

$$X_{nn} = v_{inn} - \frac{1}{F^{nn}}[\sum_{\alpha,\beta\in G} F^{\alpha\beta} v_{\alpha\beta i} + F^v v_i + F^{x_i}],$$

$X_{\alpha\beta} = v_{i\alpha\beta}$ otherwise, $Y = -v_i$ and $Z_k = -\delta_{ki}$. As $l \le n-1$, so that $(X_{\alpha\beta}) \in \mathcal{S}_{n-1}$(identity matrix) and $\tilde{X} = ((X_{\alpha\beta}), Y, Z_1, \cdots, Z_n) \in \Gamma^{\perp}_{X_F^*}$. Again by condition (3.4), we infer that

$$J_i \geq -C(\phi + \sum_{i,j\in B} |\nabla v_{ij}|).$$

Since $C \geq \sigma_l(G) + \frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)} \geq 0$, thus we obtain

$$\begin{aligned} \sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} &\leq C(\phi + \sum_{i,j\in B} |\nabla v_{ij}|) \\ &- \frac{1}{\sigma_1^3(B)} \sum_{\alpha,\beta}\sum_{i\in B} F^{\alpha\beta}(v_{ii\alpha}\sigma_1(B) - v_{ii}\sum_{j\in B} v_{jj\alpha})(v_{ii\beta}\sigma_1(B) - v_{ii}\sum_{j\in B} v_{jj\beta}) \\ &- \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta}\sum_{i,j\in B, i\neq j} F^{\alpha\beta} v_{ij\alpha}v_{ij\beta} - \frac{2}{\sigma_1^3(B)}\sum_{\alpha,\beta}\sum_{i\in B} F^{\alpha\beta} v_{ii}\sigma_1(B|i) v_{ii\alpha}v_{ii\beta}. \end{aligned} \tag{3.22}$$

The final stage of the proof is to control the term $\sum_{i,j\in B} |\nabla v_{ij}|$ in (3.22) by the rest terms on the right hand side. Let's set

$$V_{i\alpha} = v_{ii\alpha}\sigma_1(B) - v_{ii}\Big(\sum_{j\in B} v_{jj\alpha}\Big).$$

By (3.21),

$$\sum_{\alpha,\beta} F^{\alpha\beta} V_{i\alpha}V_{i\beta} \geq \delta_0 \sum_{\alpha=1}^n V_{i\alpha}^2, \quad \sum_{\alpha,\beta} F^{\alpha\beta} v_{ij\alpha}v_{ij\beta} \geq \delta_0 \sum_{\alpha=1}^n v_{ij\alpha}^2.$$

Inserting above inequalities into (3.22), we then obtain

$$\begin{aligned} \sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} &\leq C(\phi + \sum_{i,j\in B} |\nabla v_{ij}|) - \frac{\delta_0}{\sigma_1^3(B)} \sum_{\alpha=1}^n\sum_{i\in B} V_{i\alpha}^2 \\ &-\frac{\delta_0}{\sigma_1(B)}\sum_{\alpha=1}^n \sum_{i,j\in B\ i\neq j} |v_{ij\alpha}|^2 - \frac{2\delta_0}{\sigma_1^3(B)}\sum_{\alpha=1}^n\sum_{i\in B} v_{ii}\sigma_1(B|i) v_{ii\alpha}^2. \end{aligned} \tag{3.23}$$

The key differential inequality (3.5) is the consequence of (3.23) and the following lemma.

Lemma 3.3. *There is a constant C depending only on $n, \|v\|_{C^2}$ and $\frac{1}{\sigma_l(G)}$, such that for any constant $D>0$*

$$\sum_{i,j\in B}|\nabla v_{ij}| \leq C(1+\frac{2}{\delta_0}+D)(\phi+|\nabla\phi|)+\sum_{\alpha=1}^{n}[\frac{\delta_0}{2}\frac{\sum_{i,j\in B\ i\neq j}|v_{ij\alpha}|^2}{\sigma_1(B)}+\frac{C}{D}\frac{\sum_{i\in B}V_{i\alpha}^2}{\sigma_1^3(B)}]. \tag{3.24}$$

Proof of Lemma 3.3. We will use a trick devised in [14]. We break write

$$\sum_{i,j\in B}|\nabla v_{ij}| = \sum_{i,j\in B,\ i\neq j}|\nabla v_{ij}|+\sum_{i\in B}|\nabla v_{ii}|$$

If $i\neq j$, for any $A>0$, the Cauchy-Schwarz inequality yields

$$|v_{ij\alpha}| \leq 2\delta_0^{-1}\sigma_1(B)+\frac{\delta_0}{2}\frac{|v_{ij\alpha}|^2}{\sigma_1(B)} \leq C\frac{2}{\delta_0}\phi+\frac{\delta_0}{2}\frac{|v_{ij\alpha}|^2}{\sigma_1(B)}.$$

What left are the linear terms involving $v_{ii\alpha},\ i\in B$, we need the help of the second term on right hand side of (3.23) and ϕ_α. It follows from Lemma 2.4 that

$$\phi_\alpha = O(\phi)+\sum_{i\in B}(\sigma_l(G)+\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)})v_{ii\alpha}. \tag{3.25}$$

Let us now fix $\alpha\in\{1,2,\cdots,n\}$, set

$$P=\{i\in B|\ v_{ii\alpha}>0\},\ N=\{i\in B|\ v_{ii\alpha}<0\},\ R=\{i\in B|\ v_{ii\alpha}=0\}.$$

We consider two separate cases.

Case 1. Either $P=\emptyset$ or $N=\emptyset$. In this case, $v_{ii\alpha}$ has the same sign for all $i\in B$. We can derive easily

$$|v_{ii\alpha}| = O(\phi+|\phi_\alpha|). \tag{3.26}$$

Case 2. $P\neq\emptyset,\ N\neq\emptyset$. We may assume

$$\sum_{i\in P}v_{ii} \geq \sum_{j\in N}v_{jj},$$

by reversing the direction of ∂_{x_α} if necessary, since we only need to control $|v_{ii\alpha}|$. It follows from (3.25) that, for $i\in P$,

$$v_{ii\alpha} \leq \sum_{k\in P}v_{kk\alpha} \leq \frac{1}{\sigma_l(G)}O(\phi+|\phi_\alpha|)-C\sum_{j\in N}v_{jj\alpha},$$

for some positive constant C under control. At this point, we have switched the estimation of $v_{ii\alpha},\ i\in P$ to the estimation of $-v_{jj\alpha},\ j\in N$.

Claim: If $P \neq \emptyset,\ N \neq \emptyset,\ \sum_{i\in P} v_{ii} \geq \sum_{j\in N} v_{jj}$, we have

$$\Big(\sum_{j\in N} v_{jj\alpha}\Big)^2 \leq \frac{4n^2}{\sigma_1^2(B)} \sum_{i\in B} V_{i\alpha}^2.$$

If the **Claim** is true, we get for all $k \in N$,

$$\begin{aligned} -v_{kk\alpha} &\leq -\sum_{j\in N} v_{jj\alpha} \\ &\leq D\sigma_1(B) + \frac{\Big(\sum_{j\in N} v_{jj\alpha}\Big)^2}{D\sigma_1(B)} \\ &\leq CD\phi + \frac{4n^2}{D}\frac{1}{\sigma_1^3(B)} \sum_{i\in B} V_{i\alpha}^2. \end{aligned} \tag{3.27}$$

which can be controlled by the 3rd term in (3.23) if we choose the constant D large enough. Consequently we can control terms involving $v_{ii\alpha},\ i \in P$. We now validate the **Claim**.

Proof of **Claim**. We first have by the Cauchy-Schwarz inequality

$$\Big(\sum_{i\in N} V_{i\alpha}\Big)^2 \leq n^2 \sum_{i\in N} V_{i\alpha}^2 \leq n^2 \sum_{i\in B} V_{i\alpha}^2.$$

It follows that from the definitions of the sets P, N, R and $V_{i\alpha}$

$$\begin{aligned} -\sum_{i\in N} V_{i\alpha} &= \sum_{i\in N}\Big(v_{ii}(\sum_{j\in N} v_{jj\alpha} + \sum_{k\in P} v_{kk\alpha}) - v_{ii\alpha}(\sum_{j\in N} v_{jj} + \sum_{j\in R} v_{jj} + \sum_{k\in P} v_{kk})\Big) \\ &= \Big(\sum_{i\in N} v_{ii}\Big)\Big(\sum_{k\in P} v_{kk\alpha}\Big) - \Big(\sum_{k\in P\cup R} v_{kk}\Big)\Big(\sum_{i\in N} v_{ii\alpha}\Big) \end{aligned} \tag{3.28}$$

Since in this case

$$\sum_{i\in N} v_{ii} \geq 0, \sum_{k\in P} v_{kk\alpha} > 0, \sum_{j\in N} v_{jj\alpha} \leq 0,$$

all the terms on the right hand side of (3.28) are nonnegative, thus we obtain

$$\Big(\sum_{i\in N} V_{i\alpha}\Big)^2 \geq \Big(\sum_{k\in P\cup R} v_{kk}\Big)^2\Big(\sum_{i\in N} v_{ii\alpha}\Big)^2 \geq \Big(\frac{1}{2}\sum_{k\in B} v_{kk}\Big)^2\Big(\sum_{i\in N} v_{ii\alpha}\Big)^2 = \frac{\sigma_1^2(B)}{4}\Big(\sum_{i\in N} v_{ii\alpha}\Big)^2.$$

The lemma is proved. □

By Lemma 3.3 and (3.23), there exist positive constants C_1, C_2 independent of ϵ, such that

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \leq C_1(\phi + |\nabla\phi|) - C_2 \sum_{i,j\in B} |\nabla v_{ij}|. \tag{3.29}$$

Taking $\epsilon \to 0$, (3.29) is proved for u. By the Strong Maximum Principle, $\phi \equiv 0$ in $\mathcal{O}$. Since Ω is flat, following the arguments in [7, 27], for any $x_0 \in \Omega$, there is a neighborhood

$\mathcal{U}$ and $(n-l)$ fixed directions $V_1, \cdots, V_{n-l}$ such that $\nabla^2 u(x) V_j = 0$ for all $1 \le j \le n-l$ and $x \in \mathcal{U}$. The proof of Theorem 3.2 is complete. $\square$

4. Condition (1.4) and discussions

We discuss the convexity condition (1.4) in this section. We write $A^{-1} = (A^{ij})$ to be the inverse matrix A^{-1} of positive definite matrix A.

Lemma 4.1. *F satisfies Condition (1.4) if and only if*

$$\begin{aligned} &\sum_{i,j,k,l=1}^{n} F^{ij,kl}(A,p,u,x) X_{ij} X_{kl} + 2 \sum_{i,j,k,l=1}^{n} F^{ij}(A,p,u,x) A^{kl} X_{ik} X_{jl} + F^{u,u} Y^2 \\ &-2 \sum_{i,j=1}^{n} F^{ij,u} X_{ij} Y - 2 \sum_{i,j,k=1}^{n} F^{ij,x_k} X_{ij} Z_k + 2 \sum_{i=1}^{n} F^{u,x_i} Y Z_i + \sum_{i,j=1}^{n} F^{x_i,x_j} Z_i Z_j \ge 0 \end{aligned} \tag{4.1}$$

for every $X = (X_{ij}) \in \mathcal{S}^n$, $Y \in \mathbb{R}$ and $Z = (Z_i) \in \mathbb{R}^n$.

Proof. We have, from the convexity of $\tilde{F}(B,u,x) = F(B^{-1}, u, p, x)$ (for each p fixed),

$$\begin{aligned} &\sum_{\alpha,\beta,\gamma,\eta=1}^{n} \tilde{F}^{\alpha\beta,\gamma\eta}(B,u,x) \tilde{X}_{\alpha\beta} \tilde{X}_{\gamma\eta} + 2 \sum_{\alpha,\beta=1}^{n} \tilde{F}^{\alpha\beta,u} \tilde{X}_{\alpha\beta} Y + \tilde{F}^{u,u} Y^2 \\ &+2 \sum_{\alpha,\beta,k=1}^{n} \tilde{F}^{\alpha\beta,x_k} \tilde{X}_{\alpha\beta} Z_k + 2 \sum_{k=1}^{n} \tilde{F}^{u,x_k} Y Z_k + \sum_{i,j=1}^{n} F^{x_i,x_j} Z_i Z_j \ge 0 \end{aligned} \tag{4.2}$$

for every $\tilde{X} \in \mathcal{S}^n$, $Y \in \mathbb{R}$, $Z = (Z_i) \in \mathbb{R}^n$ and $B \in \mathcal{S}^n_+$. A direct computation yields

$$\begin{aligned} \tilde{F}^{\alpha\beta}(B,u,x) &= -F^{ij}(B^{-1},p,u,x) B^{i\alpha} B^{j\beta}, \\ \tilde{F}^{\alpha\beta,u}(B,u,x) &= -F^{ij,u}(B^{-1},p,u,x) B^{i\alpha} B^{j\beta}, \\ \tilde{F}^{\alpha\beta,\gamma\eta}(B,u,x) &= F^{ij,kl}(B^{-1},p,u,x) B^{i\alpha} B^{j\beta} B^{k\gamma} B^{l\eta} \\ &\quad + F^{ij}(B^{-1},p,u,x)(B^{i\gamma} B^{j\beta} B^{\eta\alpha} + B^{i\alpha} B^{j\eta} B^{\beta\gamma}). \end{aligned}$$

Other derivatives can be calculated in a similar way. Substituting these into (4.2), (4.1) follows directly. $\square$

Let $Q \in \mathbb{O}_n$, we define

$$\tilde{F}_Q(A,u,x) = F(Q \begin{pmatrix} 0 & 0 \\ 0 & A^{-1} \end{pmatrix} Q^T, p, u, x)$$

for $(A,u,x) \in \mathcal{S}^{n-1}_+ \times \mathbb{R} \times \Omega$ and fixed p. Condition (1.4) implies the following condition

$$\tilde{F}_Q(A,u,x) \quad \text{is locally convex} \tag{4.3}$$

in $\mathcal{S}^{n-1}_+ \times \mathbb{R} \times \Omega$ for any fixed $n \times n$ orthogonal matrix Q.

Lemma 4.1 yields the following by approximating.

Corollary 4.2. *Let $Q \in \mathbb{O}_n$. Assume F satisfies condition (4.3), then*

$$Q^*(\tilde{X}, \tilde{X}) \geq 0, \tag{4.4}$$

for every $\tilde{X} = ((X_{ij}), Y, Z_1, \cdots, Z_n) \in \mathcal{S}_{n-1}(Q) \times \mathbb{R} \times \mathbb{R}^n$, where Q^* is defined in (3.3).

In particular, by Corollary 4.2, condition (4.3) implies (3.4). Since condition (1.4) implies (4.3), Lemma 3.1 is a consequence of Corollary 4.2.

Condition (4.3) is weaker than condition (1.4). In particular condition (4.3) is empty condition in A when $n = 1$. There is a wide class of functions which satisfy (4.4). The most important examples are σ_k and $\frac{\sigma_l}{\sigma_k}$ $(l > k)$. If g is convex and $F_1, \cdots, F_m$ are in this class, then $F = g(F_1, \cdots, F_m)$ is also in this class. In particular, if $F_1 > 0$ and $F_2 > 0$ are in the class, so is $F = F_1^\alpha + F_2^\beta$ for any $\alpha \geq 1$, $\beta \geq 1$. Another property of condition (4.3) is the following

Corollary 4.3. *If F satisfies (4.4), then so is the function $G(A) = F(A + E)$ for any nonnegative definite matrix E.*

We also have the following lemma.

Lemma 4.4. *If $n = 2$ and $F(A) \geq 0$ is symmetric and of homogeneous of degree k. If either $k \leq 0$ or $k \geq 1$, then F satisfies (4.4).*

Proof. Since $n = 2$, condition (4.4) is equivalent to $F^{\lambda_2, \lambda_2} \geq 0$. By homogeneity, we have

$$\sum_{i,j=1}^{n} F^{\lambda_i, \lambda_j} \lambda_i \lambda_j = k(k-1)F.$$

$n = 2$ and $\lambda_1 = 0$ yields $F^{\lambda_2, \lambda_2} \lambda_2^2 = k(k-1)F(0, \lambda_2) \geq 0$. □

Simple example like $u = \sum_{i=1}^n x_i^4$, $F(A) = \sigma_1(A)$ indicates that certain condition is needed in Theorem 1.1. If F is independent of x, u, one may ask if the convexity assumption of $F(A^{-1}, p)$ for A in condition (1.4) (or condition 3.4) is necessary for Theorem 1.1. As we remarked before, when $n = 1$, it is not necessary. For general $n \geq 2$, we have the following theorem.

Theorem 4.5. *Suppose $F(A, p)$ is elliptic and u is a convex solution of*

$$F(\nabla^2 u, \nabla u) = 0, \tag{4.5}$$

then $W = (\nabla^2 u)$ is either of constant rank, or its minimal rank is at least 2. In particular, if $n = 2$, then W is of constant rank.

Proof. The proof follows same lines of proof of Theorem 3.2 with the following observations: condition (4.3) was only used to control J_i defined in (3.20). Let l be the minimum rank of W. If $l = 0$, that is $G = \emptyset$, the proof of Theorem 3.2 works without any change since F is independent of (u, x) in our case. What left is the case $l = 1$, i.e., $|G| = 1$, we may assume $\alpha = n \in G$. Note that (3.19) still holds. Since $F(\nabla^2 u, \nabla u) = 0$, and

$$0 = \nabla_i F(\nabla^2 u, \nabla u) = F^{nn} u_{nni} + O(\phi + \sum_{i,j \in B} |\nabla u_{ij}|).$$

This gives

$$|u_{nni}| \leq C(\phi + \sum_{i,j \in B} |\nabla u_{ij}|).$$

Of course, the treatment of terms involving $u_{ij\beta}$ for $i, j \in B$ follows the same way as in the proof of Theorem 3.2. We can now deduce that W is of constant. Finally, if $n = 2$, the only other case is $l = 2$. In this case, W is of full rank everywhere. □

Remark 4.6. In [6], Bramscap and Lieb proved log-concavity of the first eigenfunction of Laplacian operator for bounded convex domains in $\mathbb{R}^n$ (see also [28, 10] for different proofs). In general, for a nonlinear eigenvalue problem $F(\nabla^2 v) = \lambda v$, the function $u = -\log v$ satisfies equation (4.5) if F is of homogeneous degree of one.

Remark 4.7. The above proof of Theorem 4.5 indicates that if the minimal rank of W is either 0 or 1, then the rank of $(\nabla^2 u)$ is the same everywhere. There is no structure condition imposed on F except the ellipticity condition (1.3). This observation will be used in the proof of Theorem 1.6 in the next section.

We conclude this section with the proof of Theorem 1.2. It is a consequence of the following Strong Maximum Principle for parabolic equations.

Theorem 4.8. *Suppose that the function $F \in C^{2,1}$ satisfies conditions (1.3) and (4.4) for each $t \in [0, T]$, let $u \in C^3(\Omega \times [0, T])$ is a convex solution of (1.6). For each $0 < t_0 \leq T$, if $\nabla^2 u$ attains minimum rank l at certain point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O}$ of x_0 and a positive constant C independent of ϕ (defined in (2.2)), such that for t close to t_0, $\sigma_l(u_{ij}(x, t)) > 0$ for $x \in \mathcal{O}$, and*

$$\sum_{\alpha,\beta} F^{\alpha\beta} \phi_{\alpha\beta}(x, t) - \phi_t(x, t) \leq C(\phi(x, t) + |\nabla \phi(x, t)|), \quad \forall x \in \mathcal{O}. \tag{4.6}$$

Consequently, the rank of $\nabla^2 u(x, t)$ is constant for every fixed $t > 0$ and it is non-decreasing. For each $0 < t \leq T$, $x_0 \in \Omega$, there exist a neighborhood $\mathcal{U}$ of x_0 and $(n - l(t))$ fixed directions $V_1, \cdots, V_{n-l(t)}$ such that $\nabla^2 u(x, t) V_j = 0$ for all $1 \leq j \leq n - l(t)$ and

$x \in \mathcal{U}$. Furthermore, for any t_0, there is $\delta > 0$, such that the null space of $\nabla^2 u(x,t)$ is parallel for $(x,t) \in \mathcal{O} \times (t_0, t_0+\delta)$.

Proof of Theorem 4.8. The proof is similar to the proof of Theorem 3.2, here we will use the Strong Maximum Principle for parabolic equations.

Since $u \in C^3$, and the assumption on F, $u \in C^4$ automatically. Suppose $(\nabla^2 u(x,t_0))$ attains minimal rank l at some point $x_0 \in \Omega$. We may assume $l \le n-1$, otherwise there is nothing to prove. By continuity, $\sigma_l(u_{ij}(x,t)) > 0$ in a neighborhood of (x_0,t_0). We want to show (4.6).

With $u_t = F(\nabla^2 u, \nabla u, u, x, t)$, using the same notations as in the proof of Theorem 3.2, equation (3.12) becomes

$$\begin{aligned}
&\sum_{\alpha\beta} F^{\alpha\beta} v_{\alpha\beta ij} + \sum_{\alpha\beta} v_{\alpha\beta i}\Big(\sum_{\gamma\eta} F^{\alpha\beta,\gamma\eta} v_{\gamma\eta j} + \sum_k F^{\alpha\beta,q_k} v_{kj} + F^{\alpha\beta,v} v_j + F^{\alpha\beta,x_j}\Big)\\
&+\sum_k F^{q_k} v_{kij} + \sum_{k\alpha\beta} v_{ki}\Big(\sum_{\alpha\beta} F^{q_k,\alpha\beta} v_{\alpha\beta j} + \sum_l F^{q_k,q_l} v_{lj} + F^{q_k,v} v_j + F^{q_k,x_j}\Big)\\
&+F^v v_{ij} + v_i\Big(\sum_{\alpha\beta} F^{v,\alpha\beta} v_{\alpha\beta j} + \sum_l F^{v,q_l} u_{lj} + F^{v,v} v_j + F^{v,x_j}\Big)\\
&+\sum_{\alpha\beta} F^{x_i,\alpha\beta} v_{\alpha\beta j} + \sum_k F^{x_i,q_k} v_{kj} + F^{x_i,v} v_j + F^{x_i,x_j} = O(\phi) + v_{ij,t},
\end{aligned} \tag{4.7}$$

and accordingly, equation (3.13) becomes

$$\begin{aligned}
\sum F^{\alpha\beta}\phi_{\alpha\beta} &= \sum F^{\alpha\beta}\phi^{ij} v_{ij\alpha\beta} + \sum F^{\alpha\beta}\phi^{ij,km} v_{ij\alpha} v_{km\beta}\\
&= \sum F^{\alpha\beta}\phi^{ij,km} v_{ij\alpha} v_{km\beta} - \sum \phi^{ij} F^{q_k} v_{kij}\\
&\quad -\sum \phi^{ij}[F^v v_{ij} + 2\sum F^{\alpha\beta,q_k} v_{\alpha\beta i} v_{kj} + \sum F^{q_k,q_l} v_{ki} v_{lj}\\
&\quad +2\sum F^{q_k,v} v_{ki} v_j + 2\sum F^{q_k,x_j} v_{ki}]\\
&\quad -\sum \phi^{ij}[F^{\alpha\beta,\gamma\eta} v_{\alpha\beta i} v_{\gamma\eta j} + 2\sum F^{\alpha\beta,v} v_{\alpha\beta i} v_j + 2\sum F^{\alpha\beta,x_j} v_{\alpha\beta i}\\
&\quad +\sum F^{v,v} v_i v_j + 2\sum F^{v,x_j} v_j + \sum F^{x_i x_j}] + O(\phi) + \sum \phi^{ij} v_{ij,t}
\end{aligned} \tag{4.8}$$

We note that $\phi_t = \sum \phi^{ij} v_{ij,t}$, equation (4.8) can be written as

$$\begin{aligned}
\sum F^{\alpha\beta}\phi_{\alpha\beta} - \phi_t &= \sum F^{\alpha\beta}\phi^{ij,km} v_{ij\alpha} v_{km\beta} - \sum \phi^{ij} F^{q_k} v_{kij}\\
&\quad -\sum \phi^{ij}[F^v v_{ij} + 2\sum F^{\alpha\beta,q_k} v_{\alpha\beta i} v_{kj} + \sum F^{q_k,q_l} v_{ki} v_{lj}\\
&\quad +2\sum F^{q_k,v} v_{ki} v_j + 2\sum F^{q_k,x_j} v_{ki}]\\
&\quad -\sum \phi^{ij}[F^{\alpha\beta,\gamma\eta} v_{\alpha\beta i} v_{\gamma\eta j} + 2\sum F^{\alpha\beta,v} v_{\alpha\beta i} v_j + 2\sum F^{\alpha\beta,x_j} v_{\alpha\beta i}\\
&\quad +\sum F^{v,v} v_i v_j + 2\sum F^{v,x_j} v_j + \sum F^{x_i x_j}] + O(\phi)
\end{aligned} \tag{4.9}$$

Now the right hand side of (4.9) is the same as the right hand side of (3.13). The same analysis in the proof of Theorem 3.2 for the right hand side of equation (3.13) yields

$$\sum F^{\alpha\beta}\phi_{\alpha\beta}(x,t)-\phi_t(x,t)\le C_1(\phi(x,t)+|\nabla\phi(x,t)|)-C_2\sum_{i,j\in B}|\nabla v_{ij}|. \tag{4.10}$$

We now $\nabla^2 u(x,t)$ is of constant rank $l(t)$ for each $t>0$. Since wΩ is flat, by the arguments in [7, 27], for each $0<t\le T$, $x_0\in\Omega$, there exist a neighborhood $\mathcal{U}$ of x_0 and $(n-l(t))$ fixed directions $V_1,\cdots,V_{n-l(t)}$ such that $\nabla^2 u(x,t)V_j=0$ for all $1\le j\le n-l(t)$ and $x\in\mathcal{U}$. Now back to (4.10), we have $\sum_{i,j\in B}|\nabla u_{ij}(x,t)|\equiv 0$, therefore the null space of $\nabla^2 u$ is parallel. □

Remark 4.9. Tracing back to our proofs, for Theorem 1.1, we only need locally convexity condition in (1.4) near solution u at the points where some of eigenvalues of $\nabla^2 u$ are small. For solution u of (1.2), we let

$$\mathcal{D}_{u(x)}=\{r \text{ diagonal}|\quad r=Q(\nabla^2 u(x))Q^T\quad\text{for some } Q\in O(n)\}. \tag{4.11}$$

For each $\delta>0$, set $I^{\delta}_{u(x)}=\{s|\quad |s-u(x)|\le\delta\}$, and

$$\tilde{D}^{\delta}_{u(x)}=\{A|\quad \|A^{-1}-r\|\le\delta,\ \text{for some } r\in\mathcal{D}_{u(x)}\}.$$

The condition (1.4) in Theorem 1.1 can be replaced by: there is $\delta>0$ and for $p=Q\nabla u(x)$ $(Q\in O(n))$,

$$F(A^{-1},p,u,x)\quad\text{is locally convex in } (A,u,x) \text{ in } \tilde{D}^{\delta}_{u(x)}\times I^{\delta}_{u(x)}\times\mathcal{O}. \tag{4.12}$$

Similarly, for condition (1.5) and condition (4.3) are only needed to be valid for (A,u,x) in $\tilde{D}^{\delta}_{u(x)}\times I^{\delta}_{u(x)}\times\mathcal{O}$ for each t. We also remark that regularity assumptions on u and F in Theorem 1.2 and Theorem 4.8 can be reduced to be C^2.

5. Geometric applications

We discuss geometric nonlinear differential equations in this section.

Proposition 5.1. *Suppose $F(A,X,\vec{n},t)$ is elliptic in A and satisfies condition (4.4) for each fixed $\vec{n}\in\mathbb{S}^n$, $t\in[0,T]$ for some $T>0$. Let $M(t)$ be oriented immersed connect hypersurface in $\mathbb{R}^{n+1}$ with a nonnegative definite second fundamental form $h(t)$ satisfying equation (1.9), then $h(t)$ is of constant rank for each $t\in(0,T]$. Moreover, if let $l(t)$ be the minimal rank of $h(t)$, then $l(s)\le l(t)$ for all $0<s\le t\le T$ and the null space of h is parallel for each t.*

We note that Theorem 1.5 follows directly from Proposition 5.1 (since equation (1.10) is a special case of equation (1.9) by making M independent of t) and a splitting theorem for complete hypersurface in $\mathbb{R}^{n+1}$.

Proof of Proposition 5.1. For $\epsilon > 0$, let $W = (g^{im}h_{mj} + \epsilon\delta_{ij})$, where $h = (h_{ij})$ the second fundamental form and (g_{ij}) the first fundamental form of $M(t)$, and let $l(t)$ be the minimal rank of $h(t)$. For a fixed $t_0 \in (0, T)$, let $x_0 \in M$ such that $h(t_0)$ attains minimal rank at x_0. Set $\phi(x,t) = \sigma_{l+1}(W(x,t)) + \frac{\sigma_{l+2}}{\sigma_{l+1}}(W(x,t))$. ϕ is in $C^{1,1}$ by result of section 2. We want establish that in a small neighborhood of (x_0, t_0), there are constants C_1, C_2 independent of ϵ such that

$$F^{ij}\phi_{ij} - \phi_t \le C_1\phi + C_2|\nabla\phi|. \tag{5.1}$$

The proposition follows from (5.1) and the Strong Maximum Principle for parabolic equations by taking $\epsilon \to 0$.

We work on $W = (h_{ij} + \epsilon g_{ij})$ in place of Hessian (v_{ij}) in the proof of Theorem 3.2. We set position vector $X = (X^1, \cdots, X^{n+1})$. (5.1) can be proved using the arguments in the proofs of Theorem 3.2 and Theorem 1.2 and the Gauss equation, Codazzi equation and the Weingarten equation for hypersurfaces. We note that under (1.9), the Weingarten form $h^i_j = g^{im}h_{mj}$ satisfies equation

$$\partial_t h^i_j = \nabla^i\nabla_j F + F(h^2)^i_j, \tag{5.2}$$

where $h^2 = (h^i_l h^l_j)$.

The same arguments in the proof of Theorem 3.2 can carry through some modifications to get parabolic version of (3.12) using (5.2). In this case, W_{ijkm} and W_{kmij} may be different. But as W is Codazzi, the commutator term can be controlled using the Ricci identity. Also, p is replaced by $\vec{n}$, we use the Gauss equation when we differentiate in p. All these terms are controlled by CW_{ii}. We notice that $W_{ii} \le \phi$ for all $i \in B$, so we have the following corresponding formula to replace (3.19),

$$\begin{aligned}
\sum F^{\alpha\beta}\phi_{\alpha\beta} - \phi_t &= O(\phi + \sum_{i,j\in B}|\nabla W_{ij}|) - \frac{1}{\sigma_1(B)}\sum_{\alpha,\beta}\sum_{i,j\in B, i\neq j} F^{\alpha\beta}W_{ij\alpha}W_{ij\beta} \\
&-\frac{2}{\sigma_1^3(B)}\sum_{\alpha,\beta}\sum_{i\in B} F^{\alpha\beta}W_{ii}\sigma_1(B|i)W_{ii\alpha}W_{ii\beta} \\
&-\frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta}\sum_{i\in B} F^{\alpha\beta}(W_{ii\alpha}\sigma_1(B) - W_{ii}\sum_{j\in B} v_{jj\alpha})(W_{ii\beta}\sigma_1(B) - W_{ii}\sum_{j\in B} v_{jj\beta}) \\
&-\sum_{i\in B}[\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}][\sum_{\alpha,\beta,\gamma,\eta\in G} F^{\alpha\beta,\gamma\eta}(\Lambda)W_{i\alpha\beta}W_{i\gamma\eta} + \sum_{\alpha} F^{X^\alpha}X^\alpha_{ii} \\
&+2\sum_{\alpha\beta\in G} F^{\alpha\beta}\sum_{j\in G}\frac{1}{\lambda_j}W_{ij\alpha}W_{ij\beta} + 2\sum_{\alpha,\beta\in G}\sum_{\gamma=1}^{n+1} F^{\alpha\beta,X^\gamma}W_{i\alpha\beta}X^\gamma_i + \sum_{\gamma,\eta=1}^{n+1} F^{X^\gamma,X^\eta}X^\gamma_i X^\eta_i].
\end{aligned} \tag{5.3}$$

The term involving X_{ii} is controlled by Ch_{ii} (in turn by CW_{ii}) using the Weingarten formula. We obtain

$$\begin{aligned}
&\sum F^{\alpha\beta}\phi_{\alpha\beta}-\phi_t = O(\phi+\sum_{i,j\in B}|\nabla W_{ij}|)-\frac{1}{\sigma_1(B)}\sum_{\alpha,\beta}\sum_{i,j\in B,i\neq j}F^{\alpha\beta}W_{ij\alpha}W_{ij\beta}\\
&-\frac{2}{\sigma_1^3(B)}\sum_{\alpha,\beta}\sum_{i\in B}F^{\alpha\beta}W_{ii}\sigma_1(B|i)W_{ii\alpha}W_{ii\beta}\\
&-\frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta}\sum_{i\in B}F^{\alpha\beta}(W_{ii\alpha}\sigma_1(B)-W_{ii}\sum_{j\in B}v_{jj\alpha})(W_{ii\beta}\sigma_1(B)-W_{ii}\sum_{j\in B}v_{jj\beta})\\
&-\sum_{i\in B}[\sigma_l(G)+\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)}][\sum_{\alpha,\beta,\gamma,\eta\in G}F^{\alpha\beta,\gamma\eta}(\Lambda)W_{i\alpha\beta}W_{i\gamma\eta}\\
&+2\sum_{\alpha\beta\in G}F^{\alpha\beta}\sum_{j\in G}\frac{1}{\lambda_j}W_{ij\alpha}W_{ij\beta}+2\sum_{\alpha,\beta\in G}\sum_{\gamma=1}^{n+1}F^{\alpha\beta,X^\gamma}W_{i\alpha\beta}X_i^\gamma+\sum_{\gamma,\eta=1}^{n+1}F^{X^\gamma,X^\eta}X_i^\gamma X_i^\eta].
\end{aligned}\tag{5.4}$$

The right hand side of (5.4) is the same as in (3.19), the analysis in the proof of Theorem 3.2 can be used to show the right hand side of (5.4) can be controlled by $\phi+|\nabla\phi|-C\sum_{i,j\in B}|\nabla W_{ij}|$. The theorem follows the same argument as in the end of the proof of Theorem 4.8. □

We now use Proposition 5.1 to prove Theorem 1.4. In fact, the local convexity condition on F in that theorem can be weakened to condition (4.4).

Theorem 5.2. *Suppose $F(A,X,\vec{n},t)$ is elliptic in A and satisfies condition (4.4) for each fixed $\vec{n}\in\mathbb{S}^n$, $t\in[0,T]$ for some $T>0$. Let $M(t)\subset\mathbb{R}^{n+1}$ be compact hypersurface and it is a solution of (1.9). If M_0 is convex, then $M(t)$ is strictly convex for all $t\in(0,T)$.*

Proof of Theorem 5.2. First, we may approximate M_0 by a strictly convex M_0^ϵ. By continuity, there is $\delta>0$ (independent of ϵ), such that there is a solution $M^\epsilon(t)$ to (1.9) with $M^\epsilon(0)=M_0^\epsilon$ for $t\in[0,\delta]$. We argue that $M^\epsilon(t)$ is strictly convex for $t\in[0,\delta]$. If not, there is $t_0>0$, $M^\epsilon(t)$ is strictly convex for $0\le t<t_0$, but there is one point x_0 such that $(h_{ij}(x_0,t_0))$ is not of full rank. This is contradiction to Proposition 5.1. Taking $\epsilon\to0$, we conclude that $M(t)$ is convex for all $t\in[0,\delta]$. This implies that the set t where $M(t)$ is convex is open. It is obviously closed. Therefore, $M(t)$ is convex for all $t\in[0,T]$. Again, by Proposition 5.1, $M(t)$ is strictly convex for all $t\in(0,T]$. □

Remark 5.3. If $n=2$, by Lemma 4.4, if $F(A)$ is homogeneous of degree k for either $k\ge1$ or $k\le0$, then F satisfies condition (4.4) automatically.

Let (M,g) be a Riemannian manifold (not necessary compact), a symmetric 2-tensor W is called a Codazzi tensor if w_{ijk} is symmetric with respect to indices i,j,k in local orthonormal frames. One of the important example of the Codazzi tensor is the second fundamental form of hypersurfaces.

Theorem 5.4. *Let $F(A,x)$ is elliptic and $F(A^{-1},x)$ is locally convex in (A,x). Suppose (M,g) is a connected Riemannian manifold of nonnegative sectional curvature, and W is a semi-positive definite Codazzi tensor on M satisfying equation*

$$F(g^{-1}W,x)=0 \quad \text{on } M, \tag{5.5}$$

then W is of constant rank and its null space is parallel.

Proof. Since the proof is similar to the proof of Theorem 1.1, we only indicate some necessary modifications.

We use the same notations as in the proof of Theorem 1.1. As before, we set $\phi(x) = \sigma_{l+1}(W(x)) + \frac{\sigma_{l+2}(W(x))}{\sigma_{l+1}(W(x))}$ as in (2.2). As before, we want to establish corresponding differential inequality (3.5) in this case for the Codazzi tensor W. We note that all the analysis in Section 3 carry through without any change if we use local orthonormal frames, except the commutators of derivatives. Since W is Codazzi, we only need to take care of commutators like $W_{\alpha\alpha,\beta\beta}-W_{\beta\beta,\alpha\alpha}$. The Ricci identity states

$$W_{\alpha\alpha,\beta\beta}=W_{\beta\beta,\alpha\alpha}+R_{\alpha\beta\alpha\beta}(W_{\alpha\alpha}-W_{\beta\beta}), \tag{5.6}$$

where $R_{\alpha\beta\alpha\beta}$ the sectional curvatures of (M,g). The assumption of nonnegativity of $R_{\alpha\beta\alpha\beta}$ gives us a good sign, following the same lines of the proof of Theorem 3.2, we have the corresponding differential inequality

$$\sum_{\alpha\beta}F^{\alpha\beta}\phi_{\alpha\beta}(x)\le C_1(\phi(x)+|\nabla\phi(x)|)-\sigma_l(G)\sum_{\alpha\in G,\beta\in B}F^{\alpha\alpha}R_{\alpha\beta\alpha\beta}W_{\alpha\alpha}-C_2\sum_{i,j\in B}|\nabla W_{ij}|. \tag{5.7}$$

The strong maximum principle implies $\phi\equiv 0$ in M, so W is of constant rank l. Again, by (5.7), $\sum_{i,j\in B}|\nabla W_{ij}|\equiv 0$, so the null space of W is parallel. ☐

Proof of Theorem 1.6. We deal with case (2) of theorem first. Let $c=\min_{x\in M}W_s(x)$, where $W_s(x)$ is smallest eigenvalue of W at x. We set $\tilde W=g^{-1}(W-cg)$. Then $\tilde W$ is also a Codazzi tensor, it's rank is strictly less than n at some point, and it satisfies

$$\tilde F(\tilde W)=F(g^{-1}\tilde W+cI)=\mathit{constant}. \tag{5.8}$$

By our assumption, $c\ge 0$, it follows from Corollary 4.3 that $\tilde F$ satisfies condition (1.4). For $\phi(x)=\sigma_{l+1}(\tilde W(x))+\frac{\sigma_{l+2}(\tilde W)}{\sigma_{l+1}(\tilde W(x))}$, inequality (5.7) is valid. Therefore it follows from the proof of Theorem 3.2, $\phi\equiv 0$ in M. Now back to (5.7), the left hand side is identical

to 0, so is the right hand side. By the assumption, $R_{\alpha\beta\alpha\beta} > 0$ at some point. It follows G must be empty, that is $\tilde{W} \equiv 0$.

We now consider case (1), we follow the arguments in the proof of Theorem 4.5 and Remark 4.7. Let $\tilde{W}$ defined as before (c may not necessary nonnegative in this case). $\tilde{W}$ is a semi-positive Codazzi tensor, it's minimal rank l is strictly less than 2 at some point, and it satisfies $\tilde{F}(\tilde{W}) = F(g^{-1}\tilde{W} + cI) = 0$, and $\tilde{F}$ is elliptic. If $l = 0$, the proof for case (2) carry through without change. If $l = 1$, i.e. $|G| = 1$. At the given point, we may assume $\tilde{W}$ is diagonal and $n \in G$. Differentiate equation $\tilde{F}(\tilde{W}) = 0$, as in the proof of Theorem 4.5, we get

$$\nabla \tilde{W}_{nn} = O(\sum_{i,j\in B} \nabla \tilde{W}_{ij}).$$

Therefore, $\nabla \tilde{W}_{nn}$ can be controlled. It follows from the proof of Theorem 3.2, inequality (5.7) is valid. In turn, we get $\phi \equiv 0$ in M. As in case (2), since $R_{\alpha\beta\alpha\beta} > 0$ at some point, we must have $\tilde{W} \equiv 0$. $\square$

Remark 5.5. In spirit, our results are similar to Hamilton's strong maximum principle [19] for tensor equation

$$W_t = \Delta W + \Phi(W), \tag{5.9}$$

under the assumption that $V^T \Phi(W) V \geq 0$ for any null direction of W. Our cases are different in the setting. For example, in the case of Theorem 4.8, $W = (\nabla^2 u)$ satisfies

$$W_t = F^{ij}\nabla_i\nabla_j W + \Phi(\nabla W, W, \nabla u, u, x, t), \tag{5.10}$$

where Φ involves $\nabla W, W, \nabla u, u, x, t$. Our main analysis is to show Φ is controlled by $\phi + |\nabla\phi|$ near the null set of ϕ.

Remark 5.6. Let $\lambda_{min}(t) = \min_{x\in M(t)}\{\text{smallest eigenvalue of h(x,t)}\}$. If F in (1.9) is nonnegative and it depends only on A, using Corollary 4.3 and (5.2), by considering $W = (h^i_j(x,t)) - \lambda_{min}(s)I$, if W has zero eigenvalue at some time $t > s$, our argument in the above can show

$$\sum_{\alpha\beta} F^{\alpha\beta}\phi_{\alpha\beta}(x) - \phi_t \leq C_1\phi(x) + C_2|\nabla\phi(x)| - \sigma_l(G)\sum_{\alpha\in G,\beta\in B} F^{\alpha\alpha}R_{\alpha\beta\alpha\beta}W_{\alpha\alpha}. \tag{5.11}$$

By Theorem 1.4 the sectional curvature of $M(t)$ is strictly positive, therefore the last term in (5.11) must be vanishing, that is $W \equiv 0$. In turn, Theorem 1.4 can be strengthened as follow:

$$\lambda_{min}(t) \geq \lambda_{min}(s), \quad \forall 0 \leq s \leq t \leq T,$$

if equality holds for some $s < t_0$, then $(h^i_j(x,t)) = \lambda_{min}(s)I$ is constant for all $s \leq t$ and for all x, that is $M(t)$ is a sphere for all $t \geq s$.

Remark 5.7. Applying the same argument as in Remark 4.9, we can weaken local convexity condition on F in Theorem 1.6 and Theorem 5.4. Let

$$\mathcal{D}_{W(x)} = \{r \text{ diagonal}| \quad r = Qg^{-1}(x)W(x)Q^T \text{for some } Q \in O(n)\},$$
$$\tilde{D}^{\delta}_{W(x)} = \{A| \quad \|A^{-1} - r\| \leq \delta, \text{ for some } r \in \mathcal{D}_{u(x)}\}.$$

In this case, we only need the condition: there is $\delta > 0$,

$$F(A^{-1}, x) \text{ is locally convex in } \tilde{D}^{\delta}_{W(x)} \times \mathcal{O}\ . \tag{5.12}$$

We note that when M is compact, for given Codazzi tensor W on M, there is $\lambda > 0$ such that $\tilde{W} = \lambda g - W \geq 0$ everywhere. If $F(W)$ is concave in W, then $\tilde{F}(g^{-1}\tilde{W}) = -F(\lambda I - g^{-1}\tilde{W})$ satisfies condition (5.12).

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